1. Divisible by 19.

Suppose that $2x + 6y = 19m$. Then m is even; say $m = 2k$. Then $x + 3y = 19k$, and $5x + 15y = 5 \cdot 19k$, so $5x - 4y = 5 \cdot 19k - 19y = 19(5k - y)$. Conversely, suppose that $5x - 4y = 19n$. Then $5x + 15y = 19n + 19y = 19(n + y)$, and thus $n + y$ is a multiple of 5; say $n + y = 5r$. Then $5x + 15y = 19 \cdot 5r$, and $x + 3y = 19r$.

2. Intersections of quadratics.

The two graphs have infinitely many points in common. Since (1) factors as

$$(2x + y - 4)(x - y) = 0$$

and (2) factors as

$$(2x + y - 4)(3x - y - 2) = 0,$$

the graph of each is a pair of intersecting lines, and the two have the line $2x + y - 4 = 0$ in common. (The correct statement is that the graphs of two quadratics have at most four points of intersection or they have infinitely many.)

3. No solutions in integers.

Suppose that $m(m + 1) = n(n + 2)$. Upon adding 1 to each side we have

$$m^2 + m + 1 = (n + 1)^2.$$

But

$$m^2 < m^2 + m + 1 < (m + 1)^2;$$

i.e.,

$$m^2 < (n + 1)^2 < (m + 1)^2.$$

Thus if m is an integer, n is not.

4. Square populations.

Wew show that The 1980 population had to be $\boxed{5^2 = 25 \text{ or } 499^2 = 249001}$. Let n^2 be the 1980 population. Then $n^2 + 1000 = m^2 + 1$ and $n^2 + 2000 = r^2$ for some integers m and r. From $m^2 - n^2 = 999$ we have

$$(m - n)(m + n) = 3^3 \cdot 37.$$

Since $m - n < m + n$, we conclude that $m - n = 1, 3, 9$ or 27. The corresponding values of $m + n$ are 999, 333, 111 and 37, leading to the pairs

$$(m, n) = (500, 499),\ (168, 165),\ (60, 51),\ (32, 5),$$

respectively. Of these, the only ones for which $n^2 + 2000$ is a square are (500,499) and (32,5). Thus n must be 499 or 5, and the 1980 population had to be 499^2 or 5^2.

5. An integer valued function.

It suffices to show it for nonnegative integers, because $f(-x) = -f(x)$. Here is a proof by induction. $f(0) = 0$ is an integer. Suppose that $f(n)$ is an integer. Then

$$\begin{aligned} f(n+1) &= \frac{1}{5}(n+1)^5 + \frac{1}{3}(n+1)^3 + \frac{7}{15}(n+1) \\ &= \frac{1}{5}(n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1) + \frac{1}{3}(n^3 + 3n^2 + 3n + 1) + \frac{7}{15}(n+1) \\ &= \left(\frac{1}{5}n^5 + \frac{1}{3}n^3 + \frac{7}{15}n\right) + (n^4 + 2n^3 + 2n^2 + n) + (n^2 + n) + \left(\frac{1}{5} + \frac{1}{3} + \frac{7}{15}\right), \end{aligned}$$

which is again an integer. By induction, $f(n)$ is an integer for all integers $n \geq 0$.

6. A convergent series.

We may write

$$1 + \frac{1}{k^2 + 2k} = \frac{(k+1)^2}{k(k+2)},$$

so

$$\ln\left(1 + \frac{1}{k^2 + 2k}\right) = \ln\frac{(k+1)^2}{k(k+2)} = [\ln(k+1) - \ln k] - [\ln(k+2) - \ln(k+1)],$$

and the series telescopes:

$$\begin{aligned} \sum_{k=1}^{n} \ln\left(2 + \frac{1}{k^2 + 2k}\right) &= \sum_{k=1}^{n}\Big([\ln(k+1) - \ln k] - [\ln(k+2) - \ln(k+1)]\Big) \\ &= [\ln 2 - \ln 1] - [\ln(n+2) - \ln(n+1)] \\ &= \ln 2 - \ln\left(\frac{n+2}{n+1}\right), \end{aligned}$$

which converges to $\ln 2$ as $n \to \infty$

7. Multiplying two arithmetic sequences.

The seventh term is $\boxed{280}$. Write the first three terms of the arithmetic sequences as $a-d,\ a,\ a+d$ and $b-e,\ b,\ b+e$. Then $1216 = (a-d)(b-e) = ab - bd - ae + de$, $1360 = ab$, and $1384 = (a+d)(b+e) = ab + bd + ae + de$. Adding the first and third of these equations and dividing by 2 gives $ab + de = 1300$, whence $de = -60$, and $bd + ae = 84$. The seventh terms of the arithmetic sequences are $a + 5d$ and $b + 5e$, so the seventh term of our product sequence is

$$(a + 5d)(b + 5e) = ab + 5(bd + ae) + 25de = 1360 + (5)(84) + 25(-60) = 280.$$

8. Probability that it is divisible by 11.

The probability is $\boxed{11/126}$. There are 9! integers in which each of the nine digits occurs exactly once. We need to count those divisible by 11. Write the number

$$n = a_8 10^8 + a_7 10^7 + a_6 10^6 + a_5 10^5 + a_4 10^4 + a_3 10^3 + a_2 10^2 + a_1 10 + a_0.$$

Modulo 11,

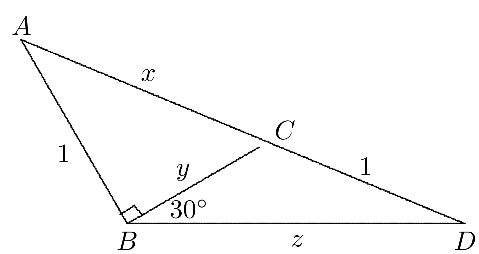
$$\begin{aligned} n &\equiv a_8(-1)^8 + a_7(-1)^7 + a_6(-1)^6 + a_5(-1)^5 + a_4(-1)^4 + a_3(-1)^3 + a_2(-1)^2 + a_1(-1) + a_0 \\ &= (-1)(a_7 + a_5 + a_3 + a_1) + (a_8 + a_6 + a_4 + a_2 + a_0), \end{aligned}$$

so n is divisible by 11 if and only if $A \equiv B \pmod{11}$, where $A = a_1 + a_3 + a_5 + a_7$ and $B = a_0 + a_2 + a_4 + a_6 + a_8$. Then we have $A + B = 45$ and we need $A - B = 11k$ for some integer k. These two conditions imply $2A = 45 + 11k$ and $2B = 45 - 11k$. We see that k must be odd and $|k| < 4$. The possible values for $2A$ then are 12, 34, 56, 78. It is easy to see that neither 6 nor 78 can occur, so A is 17 or 28. There are 9 possible sets $\{a_1, a_3, a_5, a_7\}$ with sum equal to 17, and 2 with sum 28. They are $\{1,2,5,9\}, \{1,2,6,8\}, \{1,3,4,9\}, \{1,3,5,8\}, \{1,3,6,7\}, \{1,4,5,7\}, \{2,3,4,8\}, \{2,3,5,7\}, \{2,4,5,6\}, \{4,7,8,9\}, \{5,6,8,9\}$. Thus, there are 11 choices for the set of 4 digits in A, and the other 5 digits go into B. This gives us $11(4!)(5!)$ permutations of the 9 digits yielding a multiple of 11, and the desired probability is

$$\frac{11(4!)(5!)}{9!} = \frac{11 \cdot 4 \cdot 3 \cdot 2}{6 \cdot 7 \cdot 8 \cdot 9} = \frac{11}{126}.$$

9. Length of a segment.

We show that $\boxed{AC = \sqrt[3]{2}}$. Let $x = AC$, $y = BC$, and $z = BD$. By the law of sines,



$$\frac{z}{\sin BCD} = \frac{1}{\sin 30^\circ} = 2$$

and

$$x = \frac{x}{\sin 90^\circ} = \frac{1}{\sin(180^\circ - BCD)} = \frac{1}{\sin BCD} = \frac{2}{z}.$$

By the law of cosines applied to triangle ABD we have $(1+x)^2 = 1^2 + z^2 - 2z\cos 120^\circ = 1 + z^2 + z$. Substitution of $x = 2/z$ gives $(1 + 2/z)^2 = 1 + z + z^2$; i.e., $(z+1)(z^3 - 4) = 0$. The only positive real root of this equation is $z = \sqrt[3]{4}$, and hence $x = 2/z = \sqrt[3]{2}$.

SECOND SOLUTION

Using the law of cosines appiied to angle BCD in triangle BCD we obtain

$$z^2 = 1 + y^2 + 2y^2/x. \tag{1}$$

Apply the law of cosines to angle B in the triangle ABD to get

$$(x+1)^2 = 1^2 + z^2 - 2 \cdot 1 \cdot z \cos 120^\circ = 1 + z^2 + z,$$

and thus

$$x^2 + 2x = z^2 + z. \tag{2}$$

Substitute $y^2 = x^2 - 1$ (Pythagorean Theorem) into (1) to get $z^2 = x^2 + 2(x^2-1)/x$, so

$$z^2 = x^2 + 2x - 2/x. \tag{3}$$

From (2) and (3) it follows that $z = 2/x$. Substitute $z = 2/x$ into (3), giving

$$\left(\frac{2}{x}\right)^2 = x^2 + 2x - \frac{2}{x}.$$

Now multiply both members by x^2 to get $4 = x^4 + 2x^3 - 2x$; i.e., $(x^3-2)(x+2) = 0$. As x is positive it follows that $x = \sqrt[3]{2}$

10. 2019 sums of consecutive integers .

The smallest is $\boxed{N = 3^{100} \cdot 5^4 \cdot 7 \cdot 11.}$ We note that

$$N = a + (a+1) + \cdots + (a+n) = (n+1)(2a+n)/2 \iff 2N = (n+1)(2a+n).$$

Moreover, exactly one of $n+1$ and $2a+n$ is odd, and each factor is greater than 1. Conversely, every odd divisor d of $2N$ greater than 1 uniquely determines values of n and a so that $(n+1)(2a+n) = 2N$ and d is one of $n+1$ and $(2a+n)$. We seek the smallest value of N which has exactly 2019 odd divisors greater than 1, and so has exactly 2020 odd divisors. If $N = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ where the p_j are odd primes, then the number of odd divisors of N is $(e_1+1)(e_2+1)\cdots(e_k+1)$, so we want $(e_1+1)(e_2+1)\cdots(e_k+1) = 2020 = (2)(2)(5)(101)$. There are exactly 11 factorizations of 2020: (2020), $(1010 \cdot 2)$, $(505 \cdot 4)$, $(505 \cdot 2 \cdot 2)$, $(404 \cdot 5)$, $(202 \cdot 10)$, $(202 \cdot 5 \cdot 2)$, $(101 \cdot 20)$, $(101 \cdot 10 \cdot 2)$, $(101 \cdot 5 \cdot 4)$, $(101 \cdot 5 \cdot 2 \cdot 2)$. Each corresponds to an odd number N with 2020 divisors. They are p_1^{2019}, $p_1^{1009}p_2$, $p_1^{504}p_2^3$, $p_1^{504}p_2p_3$, $p_1^{403}p_2^4$, $p_1^{201}p_2^9$, $p_1^{201}p_2^4p_3$, $p_1^{100}p_2^{19}$, $p_1^{100}p_2^9p_3$, $p_1^{100}p_2^4p_3^3$, $p_1^{100}p_2^4p_3p_4$. With $p_1 = 3$, $p_2 = 5$, $p_3 = 7$, and $p_4 = 11$, these eleven values of N are in decreasing order, and the smallest is $3^{100} \cdot 5^4 \cdot 7 \cdot 11$.