1. Multiples of 23.

Well, $3^n 2^{3n} - 1 = 3^n 8^n - 1 = 24^n - 1$, and this is divisible by $24 - 1$ because $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + 1)$ is divisible by $x - 1$ for every n.

2. The 2018th term.

We will show that $\boxed{x_{2018}^2 - y_{2018} = 2}$. We prove by induction that $y_n = x_n^2 - 2$ for all n. For $n = 0$ this is given. Suppose that $y_k = x_k^2 - 2$. Then

$$\begin{aligned} y_{k+1} &= y_k^3 - 3y_k \\ &= (x_k^2 - 2)^3 - 3(x_k^2 - 2) \\ &= x_k^6 - 6x_k^4 + 12x_k^2 - 8 - 3x_k^2 + 6 \\ &= x_k^6 - 6x_k^4 + 9x_k^2 - 2 \\ &= (x_k^3 - 3x_k)^2 - 2 \\ &= x_{k+1}^2 - 2. \end{aligned}$$

By the induction principle, $y_n = x_n^2 - 2$ for all n, and in particular, $x_{2018}^2 - y_{2018} = 2$

3. Sums of fourth powers.

A general identity of which the given equations are special cases is

$$\boxed{a^4 + b^4 + (a+b)^4 = 2(a^2 + ab + b^2)^2.}$$

This identity is readily verified by expansion. How to find it? That the left-hand members are of the form $a^4 + b^4 + (a+b)^4$ one notices rather readily. One may also notice that the right-hand members have the form $2(a^2 + ab + b^2)^2$; otherwise, on expanding

$$\frac{1}{2}(a^4 + b^4 + (a+b)^4) = a^4 + 2a^3b + 3a^2b^2 + 2ab^3 + b^4, \tag{1}$$

it is natural to compare this with

$$(a^2 + x + b^2)^2 = a^4 + 2a^2x + (x^2 + 2a^2b^2) + 2b^2x + b^4. \tag{2}$$

For the right-hand members of (1) and (2) to agree, we need

$$2a^2x + (x^2 + 2a^2b^2) + 2b^2x = 2a^3b + 3a^2b^2 + 2ab^3,$$

and one quickly sees that $x = ab$ does the job.

4. Each face occurs twice..

The probability is $\boxed{\dfrac{5^2 \cdot 7 \cdot 11}{2^8 \cdot 3^7}}$. There are 6^{12} possible outcomes of the 12 rolls. The number in which each face occurs exactly twice is

$$\binom{12}{2}\binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2} = \frac{12!}{10!2!} \cdot \frac{10!}{8!2} \cdot \frac{8!}{6!2!} \cdot \frac{6!}{4!2!} \cdot \frac{4!}{2!2!} \cdot \frac{2!}{2!} = \frac{12!}{2^6},$$

so the desired probability is

$$\begin{aligned}\frac{12!}{2^6 \cdot 6^{12}} &= \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{2^{18} \cdot 3^{12}} \\ &= \frac{2^2 \cdot 3 \cdot 11 \cdot 2 \cdot 5 \cdot 3^2 \cdot 2^3 \cdot 7 \cdot 2 \cdot 3 \cdot 5 \cdot 2^3 \cdot 3}{2^{18} \cdot 3^{12}} \\ &= \frac{2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11}{2^{18} \cdot 3^{12}} = \frac{5^2 \cdot 7 \cdot 11}{2^8 \cdot 3^7}.\end{aligned}$$

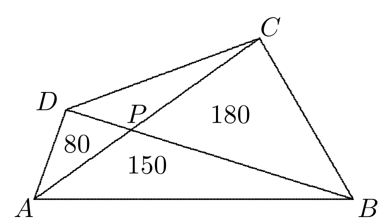
(Multiplied out this is $\dfrac{1925}{559872}$, or about 0.00344.)

5. A subset of the first 2018 positive integers.

Suppose, on the contrary, that there are no two elements of B which differ by 31. Then the set $C = B + 31 = \{b + 31 \colon b \in B\}$ has 1025 elements, none of which is in B, and at most 31 of which are greater than 2018. $(2018 + 31, 2017 + 31, \ldots 1988 + 31)$. Thus at least $1025 - 31 = 994$ elements of C are in $S \setminus B$. But then B and $S \setminus B$ together have at least $994 + 1025 = 2019$ elements, all of which are in S, contradicting that S has just 2018 elements. Thus B does have two elements which differ by 31.

6. Area of a quadrilateral.

The area of $ABCD$ is $\boxed{506}$. Let P be the point of intersection of the diagonals. In general, let (XYZ) denote the area of triangle XYZ. Let h_D be the altitude from D to AC, and h_B the altitude from B to AC. Then



$$\frac{(APD)}{(CPD)} = \frac{(AP)h_D}{(CP)h_D} = \frac{(AP)}{(CP)}, \text{ and } \frac{(ABP)}{(CBP)} = \frac{(AP)h_B}{(CP)h_B} = \frac{(AP)}{(CP)},$$

so

$$\frac{80}{(CPD)} = \frac{(APD)}{(CPD)} = \frac{(AP)}{(CP)} = \frac{(ABP)}{(CBP)} = \frac{150}{180} = \frac{5}{6}.$$

Thus $(CPD) = (80)(6/5) = 96$, and the area of the quadrilateral is $180+150+80+96 = 506$.

7. A quadratic equation with integral roots.

They are $r = -\frac{1}{7}$ and $r = 1$. We first show that these values of r work. If $r = -\frac{1}{7}$ the equation becomes

$$-\frac{1}{7}x^2 + \frac{6}{7}x - \frac{8}{7} = 0,$$

which is equivalent to $x^2 - 6x + 8 = 0$, and the roots are 2 and 4. If $r = 1$ the equation is $x^2 + 2x = 0$, which has roots 0 and -2. It remains to show that these are the only such values of r. Suppose the equation has integer roots m and n. Then

$$(x - m)(x - n) = x^2 + \left(1 + \frac{1}{r}\right)x + \left(1 - \frac{1}{r}\right),$$

from which we see that $mn = 1 - \frac{1}{r}$ and $-m - n = 1 + \frac{1}{r}$. Adding these two equations we obtain $mn - m - n = 2$, and thus

$$(m - 1)(n - 1) = mn - m - n + 1 = 3.$$

It follows that $\{m - 1, n - 1\} = \{3, 1\}$ or $\{-3, -1\}$. In the first case $\{m, n\} = \{4, 2\}$ and we get $r = -\frac{1}{7}$. In the second case $\{m, n\} = \{-2, 0\}$ and $r = 1$.

8. 2018+xy is a perfect square.

No such four integers exist. Consider any four distinct positive integers, and look at things modulo 4. Every square is either 0 or 1 mod 4 and 2018 is 2 mod 4, so we need the product of each two of our four integers to be 2 or 3 mod 4. Then we cannot have two evens among our four, for their product would be 0 mod 4. Thus at least three of the four must be odd. Some two of these three are congruent mod 4, so their product is 1 mod 4, and this added to 2018 would be 3 mod 4 and therefore not a square. Thus no four integers have the desired property.

9. A max/min problem.

We show that the maximum value is $\boxed{1/4}$. Note that $f(1/2, 1/2, 1/2) = 1/4$, and we need only show that there are no larger values. Suppose, on the contrary, that for some (x, y, z) we had $f(x, y, z) > 1/4$. Then

$$x - y^2 > \frac{1}{4}, \qquad y - z^2 > \frac{1}{4} \qquad \text{and} \qquad z - x^2 > \frac{1}{4}.$$

Adding these three inequalities, we find that

$$(x + y + z) - (x^2 + y^2 + z^2) > \frac{3}{4},$$

i.e.,

$$(x^2 - x) + (y^2 - y) + (z^2 - z) + \frac{3}{4} < 0.$$

But this implies that

$$\left(x^2 - x + \frac{1}{4}\right) + \left(y^2 - y + \frac{1}{4}\right) + \left(z^2 - z + \frac{1}{4}\right) < 0;$$

i.e., that

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 + \left(z - \frac{1}{2}\right)^2 < 0,$$

which is obviously impossible. Thus $f(x, y, z)$ takes no values greater than 1/4

10. A logarithmic inequality .

The desired inequality is equivalent to

$$a(\ln a - \ln c) + b(\ln b - \ln d) \geq 0 = (a + b) - (c + d);$$

i.e., to

$$a \ln(a/c) + b \ln(b/d) \geq (a - c) + (b - d).$$

For this it suffices to show that

$$a \ln(a/c) \geq a - c \qquad \text{and} \qquad b \ln(b/d) \geq b - d;$$

i.e., to show that

$$\frac{a}{c} \ln\left(\frac{a}{c}\right) \geq \frac{a}{c} - 1 \qquad \text{and} \qquad \frac{b}{d} \ln\left(\frac{b}{d}\right) \geq \frac{b}{d} - 1. \tag{1}$$

We show that for all $x > 0$,

$$x \ln x \geq x - 1, \tag{2}$$

and (1) follows. Let $f(x) = x \ln x - x + 1$ for $x > 0$. Then $f(1) = 0$ and $f'(x) = \ln x$, which is negative in (0,1) and positive for $x > 1$, showing that $f(x) > 0$ in (0,1) and in $(1, \infty)$. This proves (2), and therefore (1).