1. An averaging operation.

The unique solution is $\boxed{\text{m=2017}}$. We have

$$m = m * (m * 2017) = m * \left(\frac{m + 2017}{2}\right) = \frac{1}{2}\left(m + \frac{m + 2017}{2}\right) = \frac{3m + 2017}{4},$$

so $4m = 3m + 2017$ and $m = 2017$.

2. An inequality

With $k = 0$ we have $1 + 2 + 3 = 6$, and with $k = 1$, $1^3 + 2^3 + 3^3 = 36 = 6^2$. Now, with $k = 2$ we have

$$1^5 + 2^5 + 3^5 > 3^5 = 243 > 216 = 6^3.$$

If $3^{2k+1} > 6^{k+1}$, then $3^{2k+3} > 9 \cdot 6^{k+1} > 6^{k+2}$, so by induction $3^{2k+1} > 6^{k+1}$ for all $k \geq 2$, and thus

$$1^{2k+1} + 2^{2k+1} + 3^{2k+1} \geq 6^{k+1}$$

for all integers $k \geq 0$.

3. Probability the sum is odd.

The probability is $\boxed{11/21}$. The sum is odd if and only if the number of odds drawn is odd: i.e., the five balls drawn contain exactly 1 odd and 4 even or 3 odds and 2 evens or 5 odds and no evens. There are 5 odd-numbered balls and 4 even-numbered balls, so the probability is

$$\frac{\binom{5}{1}\binom{4}{4} + \binom{5}{3}\binom{4}{2} + \binom{5}{5}\binom{4}{0}}{\binom{9}{5}} = \frac{5 \cdot 1 + 10 \cdot 6 + 1 \cdot 1}{126} = \frac{66}{126} = \frac{11}{21}.$$

4. A sum of reciprocals.

There are 9 such pairs: $\{15, 210\}$, $\{16, 112\}$, $\{18, 63\}$, $\{21, 42\}$, $\{28, 28\}$, $\{13, -182\}$, $\{12, -84\}$, $\{10, -35\}$, and $\{7, -14\}$, as we shall show. Multiplying both members of the equation by $14xy$ and transposing gives the equivalent equation $xy - 14x - 14y = 0$. Now add 196 to both members and factor:

$$(x - 14)(y - 14) = 196 = 2^2 7^2.$$

The divisors of 196 are 1, 2, 4, 7, 14, 28, 49, 98, 196, and their negatives. As x and y are to be integers, so are $x-14$ and $y-14$. Thus the possible unordered pairs for $\{x-14, y-14\}$ are $\{1, 196\}$, $\{2, 98\}$, $\{4, 49\}$, $\{7, 28\}$, $\{14, 14\}$, $\{-1, -196\}$, $\{-2, -98\}$, $\{-4, -49\}$, $\{-7, -28\}$, and $\{-14, -14\}$. To each of these but the last we add 14 to each member of the pair to get a pair $\{x, y\}$ satisfying the original equation. The last pair would yield $x = y = 0$, which is not a solution. Thus we have the nine solutions given at the outset.

5. Floor-function integral.

The value is $\boxed{17 - \sum_{n=2}^{7} \sqrt[3]{n}}$. We have

$$\int_2^4 \lfloor x^3 - 6x^2 + 12x - 6 \rfloor dx = \int_2^4 \lfloor (x-2)^3 + 2 \rfloor dx = \int_2^4 \lfloor (x-2)^3 \rfloor dx + 4.$$

Make the substitution $u = x - 2$. Then $\int_2^4 \lfloor (x-2)^3 \rfloor dx = \int_0^2 \lfloor u^3 \rfloor du$.

For $0 \le u < 1$ we have $0 \le u^3 < 1$, and $\lfloor u^3 \rfloor = 0$.

For $1 \le u < \sqrt[3]{2}$ we have $1 \le u^3 < 2$, and $\lfloor u^3 \rfloor = 1$.

For $\sqrt[3]{2} < \sqrt[3]{3}$, we have $1 \le u^3 < 3$, and $\lfloor u^3 \rfloor = 2$.]

In general, for $\sqrt[3]{n} \le u < \sqrt[3]{n+1}$ we have $\lfloor u^3 \rfloor = n$, and we use this up to $n = 7$: For $\sqrt[3]{7} \le u < \sqrt[3]{8}$, $\lfloor u^3 \rfloor = 7$. Thus

$$\begin{aligned}
\int_0^2 \lfloor u^3 \rfloor du &= \int_0^1 0 du + \int_1^{\sqrt[3]{2}} 1 du + \int_{\sqrt[3]{2}}^{\sqrt[3]{3}} 2 du + \cdots + \int_{\sqrt[3]{7}}^2 7 du \\
&= 0 + 1(\sqrt[3]{2} - 1) + 2(\sqrt[3]{3} - \sqrt[3]{2}) + 3(\sqrt[3]{4} - \sqrt[3]{3}) + 4(\sqrt[3]{5} - \sqrt[3]{4}) + 5(\sqrt[3]{6} - \sqrt[3]{5}) \\
&\qquad + 6(\sqrt[3]{7} - \sqrt[3]{6}) + 7(2 - \sqrt[3]{7}) \\
&= -1 - \sqrt[3]{2} - \sqrt[3]{3} - \sqrt[3]{4} - \sqrt[3]{5} - \sqrt[3]{6} - \sqrt[3]{7} + 14.
\end{aligned}$$

Adding in the 4 gives us $17 - \sum_{n=2}^{7} \sqrt[3]{n}$.

6. A sum of 2017 terms.

We note that

$$a_n = \frac{2^n}{(2^n - 1)(2^{n+1} - 1)} = \frac{1}{2^n - 1} - \frac{1}{2^{n+1} - 1},$$

so the sum telescopes:

$$\begin{aligned}
a_1 + a_2 + \cdots + a_{2017} &= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{7}\right) + \cdots + \left(\frac{1}{2^{2017} - 1} - \frac{1}{2^{2018} - 1}\right) \\
&= 1 - \frac{1}{2^{2018} - 1} < 1.
\end{aligned}$$

7. Counting permutations.

There are $\boxed{126}$ such permutations. Because a_5 is smaller than each of the other a_k, we must have $a_5 = 1$. Now choose any four of the remaining nine numbers to occupy the first four positions. Their order is then uniquely determined, as is the order of the remaining five numbers. Thus the number of permutations of the desired type is just the number of ways to choose four numbers from nine, namely $\binom{9}{4} = 126$.

8. A limit.

The limit is $\boxed{1/2}$. Using $a - b = (a^2 - b^2)/(a + b)$ we have

$$\sqrt{x^2+1} - x = \frac{x^2+1-x^2}{\sqrt{x^2+1}+x} = \frac{1}{\sqrt{x^2+1}+x}. \tag{1}$$

Using $a - b = (a^3 - b^3)/(a^2 + ab + b^2)$ we have

$$\sqrt[3]{x^3+1} - x = \frac{x^3+1-x^3}{(x^3+1)^{2/3}+(x^3+1)^{1/3}x+x^2} = \frac{1}{(x^3+1)^{2/3}+(x^3+1)^{1/3}x+x^2}. \tag{2}$$

Subtracting (2) from (1) and multiplying by x we obtain

$$x\Big(\sqrt{x^2+1} - \sqrt[3]{x^3+1}\Big) = \frac{x}{\sqrt{x^2+1}+x} - \frac{x}{(x^3+1)^{2/3}+(x^3+1)^{1/3}x+x^2}. \tag{3}$$

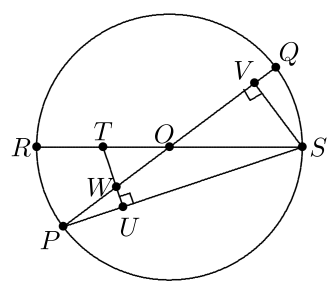
The second fraction on the right in (3) is smaller than $x/x^2 = 1/x$ for positive x, so goes to 0 as $x \to \infty$. The first term may be rewritten

$$\frac{1}{\sqrt{1+\frac{1}{x^2}}+1},$$

so has limit $1/2$. Thus

$$\lim_{x\to\infty} x\Big(\sqrt{x^2+1} - \sqrt[3]{x^3+1}\Big) = \frac{1}{2}.$$

9. Equal products of side lengths.



What we have to show is equivalent to $\dfrac{PU}{PV} = \dfrac{OT}{PS}$. Let W be the intersection of the segment TU with the diameter PQ. From the similarity of the right triangles PUW and PVS we know that $\dfrac{PU}{PV} = \dfrac{PW}{PS}$, so it would suffice to show that $PW = OT$, half the radius. This, in turn, would follow if we show $OW = OT$; i.e., that the triangle OWT is isosceles, with $\angle OTW = \angle OWT$. This we can do: $\angle OTW = 90^\circ - \angle TSU = 90^\circ - \angle OPS = \angle PWU = \angle OWT$, establishing the claim.

10. Infinitely many terms divisible by 2017.

We have $a_3 = 45^2 - 8 = 2017$. Now, mod 2017, only finitely many ordered pairs (a_n, a_{n+1}) can occur, so eventually there is a repetion: $(a_{n+r}, a_{n+r+1}) = (a_n, a_{n+1})$ for some n and r. But the recursion is reversible: $a_n = a_{n+1}^2 - a_{n+2}$. This implies that each pair (a_n, a_{n+1}) has a uniquely determined predecessor (a_{n-1}, a_n), and thus the sequence is fully periodic modulo 2017. Hence the initial pair $(8, 45)$ followed by $(45, 2017) \equiv (45, 0) \pmod{2017}$ occurs somewhere within each cycle of length r. Thus, infinitely many terms are divisible by 2017.