1. A geometric progression.

The unique solution is $\boxed{x = 1/3}$. We have

$$\frac{8^{2x}}{4^{3x-1}} = \frac{16}{8^{2x}},$$

so

$$\frac{2^{6x}}{2^{6x-2}} = \frac{2^4}{2^{6x}}; \quad 2^2 = 2^{4-6x}.$$

Then $6x = 2$, and $x = 1/3$.

2. A 2016 evaluation

We show that $\boxed{x^3 + 1/x^3 = 2015\sqrt{2018}}$. We have

$$\left(x + \frac{1}{x}\right)^2 = x^2 + 2 + \frac{1}{x^2} = 2018,$$

so $x + 1/x = \sqrt{2018}$. Then

$$\left(x + \frac{1}{x}\right)^3 = 2018\sqrt{2018} = x^3 + 3x + \frac{3}{x} + \frac{1}{x^3} = x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right) = x^3 + \frac{1}{x^3} + 3\sqrt{2018},$$

and $x^3 + 1/x^3 = 2018\sqrt{2018} - 3\sqrt{2018} = 2015\sqrt{2018}$.

3. Some cubic polynomials.

They are $\boxed{x^3,\ x^3 - ax^2 \text{ for arbitrary } a \neq 0, \text{ and } x^3 + x^2 - x - 1}$.

From $x^3 - ax^2 + bx - c = (x-a)(x-b)(x-c)$ we have

$$a + b + c = a, \tag{1}$$

$$ab + ac + bc = b, \tag{2}$$

and

$$abc = c. \tag{3}$$

From (1) we see that $b + c = 0$, so (2) reduces to $bc = b$, which implies that $b = 0$ or $c = 1$. If $b = 0$, then $c = 0$ as well, and all three conditions are satisfied. This gives us x^3 and $x^3 - ax^2$ for arbitrary $a \neq 0$. If $b \neq 0$ then $c = 1$, and (1) implies $b = -1$ and (3) implies $a = -1$. This gives us the solution $x^3 + x^2 - x - 1$. It is easy to check that each of x^3, $x^3 - ax^2$ and $x^3 + x^2 - x - 1$ satisfies the specified condition.

4. A sum of squared sines.

The sum is 45.5. Pair the terms up as follows:

$$\begin{aligned} S = &\sin^2 0^\circ + \sin^2 90^\circ \\ &+ \sin^2 1^\circ + \sin^2 89^\circ \\ &+ \sin^2 2^\circ + \sin^2 88^\circ \\ &+ \cdots \\ &+ \sin^2 44^\circ + \sin^2 46^\circ \\ &+ \sin^2 45^\circ. \end{aligned}$$

Because $\sin^2(90^\circ - \theta) = \cos^2\theta$, each of the pairs $\sin^2 n^\circ + \sin^2(90-n)^\circ = \sin^2 n^\circ + \cos^2 n^\circ = 1$ for $0 \le n \le 44$. Thus we have

$$S = 45 + \sin^2 45^\circ = 45 + \frac{1}{2}.$$

5. Lots of powers of 2.

Because $|1-\sqrt{3}| < 1$, we have $0 < (1-\sqrt{3})^{2n} < 1$ for all positive integers n. Note that $(1+\sqrt{3})^{2n} = r + s\sqrt{3}$ for some integers r and s, and that $(1-\sqrt{3})^{2n} = r - s\sqrt{3}$. Thus

$$(1+\sqrt{3})^{2n} + (1-\sqrt{3})^{2n} = 2r = \lceil (1+\sqrt{3})^{2n} \rceil.$$

Now, $(1 \pm \sqrt{3})^2/2 = 2 \pm \sqrt{3}$, so $(1 \pm \sqrt{3})^{2n}/2^n = (2 \pm \sqrt{3})^n = t \pm u\sqrt{3}$ for some integers t and u. Then

$$\frac{\lceil (1+\sqrt{3})^{2n} \rceil}{2^n} = \frac{2r}{2^n} = \frac{(1+\sqrt{3})^{2n}}{2^n} + \frac{(1-\sqrt{3})^{2n}}{2^n} = 2t,$$

showing that $\lceil (1+\sqrt{3})^{2n} \rceil / 2^{n+1} = t$, an integer.

6. Sides of a triangle.

The side lengths are $\boxed{24/\sqrt{15},\ 16/\sqrt{15} \text{ and } 32/\sqrt{15}}$. Let a, b, c be the sides corresponding to altitudes 4, 6 and 3, respectively. Then $4a = 6b = 3c$, each being twice the area of te triangle. The area of the triangle is also given by $\sqrt{s(s-a)(s-b)(s-c)}$, where

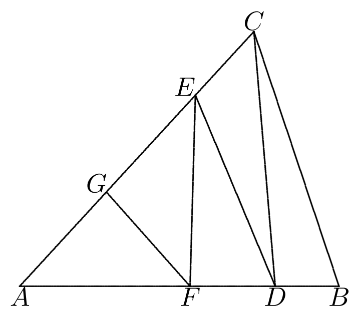
$$s = \frac{1}{2}(a+b+c) = \frac{1}{2}\left(a + \frac{2}{3}a + \frac{4}{3}a\right) = \frac{3}{2}a.$$

Equating the two expressions for the area we have

$$2a = \sqrt{\left(\frac{3}{2}a\right)\left(\frac{1}{2}a\right)\left(\frac{5}{6}a\right)\left(\frac{1}{6}a\right)} = \frac{a^2}{12}\sqrt{15}.$$

Thus $a = 24/\sqrt{15}$, $b = 2a/3 = 16/\sqrt{15}$, and $c = 4a/3 = 32/\sqrt{15}$.

7. Five triangles of equal area.



We show that $\boxed{FD = 8}$. Triangle ADC is 4/5 of the area of ABC and they have the same altitude from the base along AB, so $AD = (4/5)(AB) = 24$. Similarly triangle FED has 1/3 the area of AED and they have the same altitude from the base on AD. Therefore $FD = (1/3)(AD) = (1/3)(24) = 8$.

8. Divisible by 2^{2016}.

$\boxed{\text{Yes}}$, there is such an integer. We show, by induction, that for every positive integer n, there is an n-digit integer N, each digit of which is 6 or 7, which is divisible by 2^n. With $n = 1$ we take $N = 6$. We now show that if N is a k-digit number, each digit 6 or 7, which is divisible by 2^k, then one of the two numbers obtained by putting a 6 or a 7 in front of N will be divisible by 2^{k+1}. For example, from 6 we go to 76, divisible by 4, to 776, divisible by 8.

So, let N be a k-digit number, each digit 6 or 7, and divisible by 2^k. We may write $N = 2^k \cdot r$. The integer obtained by putting 6 or 7 in front of N is $2^k \cdot r + 6 \cdot 10^k$ or $2^k \cdot r + 7 \cdot 10^k$, respectively. If r is even, then $2^k \cdot r + 6 \cdot 10^k$ is divisible by 2^{k+1}. If r is odd, then $2^k \cdot r + 7 \cdot 10^k = 2^k \cdot r + 2^k \cdot 7 \cdot 5^k = 2^k(r + 7 \cdot 5^k)$, which is divisible by 2^{k+1} because $r + 7 \cdot 5^k$ is even. By induction, the claim is proved.

9. All terms integers?

We show that every term is an integer. Clear the recursion of fractions to get $a_n a_{n-2} = a_{n-1}^2 + 13$, and then also $a_{n+1}a_{n-1} = a_n^2 + 13$. Subtracting the first of these from the second gives $a_{n+1}a_{n-1} - a_n a_{n-2} = a_n^2 - a_{n-1}^2$, which we rewrite $a_n^2 + a_n a_{n-2} = a_{n-1}^2 + a_{n-1}a_{n+1}$. With a change of notation,

$$a_k(a_k + a_{k-2}) = a_{k-1}(a_{k-1} + a_{k+1}).$$

Consider this last equation now for $3 \leq k \leq n$:

$$\begin{aligned}
a_3(a_3 + a_1) &= a_2(a_2 + a_4) \\
a_4(a_4 + a_2) &= a_3(a_3 + a_5) \\
a_5(a_5 + a_3) &= a_4(a_4 + a_6) \\
&\vdots \\
a_{n-1}(a_{n-1} + a_{n-3}) &= a_{n-2}(a_{n-2} + a_n) \\
a_n(a_n + a_{n-2}) &= a_{n-1}(a_{n-1} + a_{n+1}).
\end{aligned}$$

Equating the product of the left-hand members with the product of the right-hand members we obtain

$$(a_3a_4\cdots a_n)(a_3+a_1)(a_4+a_2)\cdots(a_{n-1}+a_{n-3})(a_n+a_{n-2})$$
$$=(a_2a_3\cdots a_{n-1})(a_2+a_4)(a_3+a_5)\cdots(a_{n-2}+a_n)(a_{n-1}+a_{n+1}).$$

Canceling common factors leaves

$$a_n(a_3+a_1)=a_2(a_{n-1}+a_{n+1}).$$

The initial terms are $a_1=1$, $a_2=7$ and $a_3=62$, so $63a_n=7a_{n-1}+7a_{n+1}$. Thus, $a_{n+1}=9a_n-a_{n-1}$, from which it is clear that all terms are integers.

10. An inequality.

We have

$$a^3+b^3=(a+b)(a^2-ab+b^2)=(a+b)((a-b)^2+ab)\geq(a+b)(ab).$$

Similarly, $b^3+c^3\geq(b+c)(bc)$ and $c^3+a^3\geq(c+a)(ca)$. Adding these three inequalities, we obtain

$$2(a^3+b^3+c^3)\geq a^2b+ab^2+b^2c+bc^2+c^2a+ca^2.$$

Then

$$a^3+b^3+c^3\geq a^2\left(\frac{b+c}{2}\right)+b^2\left(\frac{a+c}{2}\right)+c^2\left(\frac{a+b}{2}\right)$$
$$\geq(a^2)\sqrt{bc}+(b^2)\sqrt{ac}+(c^2)\sqrt{ab},$$

where we have used the AM,GM inequality in the last step.