1. Equation of a circle.

The equation of the circle is

$$\left(x - \frac{3}{2}\right)^2 + \left(y - \frac{7}{6}\right)^2 = \frac{65}{18}.$$

Let $B = (2, b)$. Then the center is $(\frac{3}{2}, \frac{3+b}{2})$, and equating distances from the center to $(0, 0)$ and $(1, 3)$ we get, on simplification, $b = -\frac{2}{3}$. Thus the center is $(\frac{3}{2}, \frac{7}{6})$ and the radius is $\frac{\sqrt{130}}{6}$, giving the equation as asserted above.

2. Simultaneous equations.

The unique solution is $\boxed{x = 29.7,\ y = 13.4}$. Let $m = \lfloor x \rfloor$ and $n = \lfloor y \rfloor$. From the second equation we see that the fractional part of x is .7, so $x = m + .7$. Writing $y = n + h$, where $0 \le h < 1$, we have from the first equation that $m + .7 + h = 30.1$; $m + h = 29.4$, so $m = 29$ and $h = .4$. Thus $x = 29.7$. Then the second equation yields $29.7 + 29 + n = 71.7$; i.e., $n = 13$, whence $y = 13.4$.

3. 2014 concentric circles .

The ratio is $\boxed{2013/4028}$. The area of the black region is

$$\pi[1^2 + (3^2 - 2^2) + (5^2 - 4^2) + \cdots + (2013^2 - 2012^2)],$$

so the ratio in question is

$$\frac{1 + (3-2)(3+2) + (5-4)(5+4) + \cdots + (2013-2012)(2013+2012)}{2014^2}$$
$$= \frac{1 + (2+3) + (4+5) + \cdots + (2012+2013)}{2014^2}$$
$$= \frac{(2013)(2014)}{2(2014)^2} = \frac{2013}{4028}.$$

4. Cubic polynomial game.

Adolf wins by choosing (for example) $a = 1$ and $c > b^2/3$. For then,

$$f'(x) = 3x^2 + 2bx + c = 3\left(x + \frac{b}{3}\right)^2 + \left(c - \frac{b^2}{3}\right),$$

which is positive for all x. This implies that $f(x)$ is strictly increasing so has just one real zero, and therefore Adolf wins.

5. Digit sum of the cube.

There are only finitely many such numbers. We'll show that all of them are smaller than 100. Suppose that the integer n in decimal form has k digits. Then $10^{k-1} \le n < 10^k$, and $n^3 < 10^{3k}$, so n^3 has at most $3k$ digits. Then $\mathrm{SDD}(n^3) \le 9(3k) = 27k$, where SDD represents sum of decimal digits. Thus, if $27k < 10^{k-1}$, we would have

$$\mathrm{SDD}(n^3) \le 27k < 10^{k-1} \le n,$$

and $\mathrm{SDD}(n^3) = n$ would be impossible. We now show by induction that

$$27k < 10^{k-1} \quad \text{whenever } k \ge 3. \tag{1}$$

For $k = 3$ we have $27k = 27 \cdot 3 = 81 < 10^2 = 10^{k-1}$. Suppose that k is an integer, $k \ge 3$, such that $27k < 10^{k-1}$. Then

$$27(k+1) = 27k+27 < 10^{k-1}+27 < 10^{k-1}+10^2 \le 10^{k-1}+10^{k-1} = 2\cdot 10^{k-1} < 10\cdot 10^{k-1} = 10^k.$$

By induction, (1) follows, and therefore no integer n of 3 or more digits has $\mathrm{SDD}(n^3) = n$.

6. No real zeros.

One standard way to simplify a sum of this form is to add $xP(x)$ to $P(x)$:

$$P(x) = x^{50} - 2x^{49} + 3x^{48} - \cdots - 50x + 51;$$

$$xP(x) = x^{51} - 2x^{50} + 3x^{49} - 4x^{48} + \cdots - 50x^2 + 51x.$$

Adding these two equations we obtain

$$\begin{aligned}(1+x)P(x) &= x^{51} - x^{50} + x^{49} - x^{48} + \cdots - x^2 + x + 51\\ &= x\left(\frac{x^{51}+1}{x+1}\right) + 51 \quad \text{for } x \ne -1.\end{aligned}$$

From this it is clear that $P(x) > 0$ for $x \ge 0$. For $x < 0$, every term of $P(x)$ is positive, so $P(x) > 0$ here as well. Thus $P(x) > 0$ for all real x.

7. Not both rational.

Rewrite the equation in the form $a^4b^4 - 2a^6 - 2b^6 + 4a^2b^2 = 0$ and then factor to obtain $(a^4 - 2b^2)(b^4 - 2a^2) = 0$. Thus either $a^4 = 2b^2$ or $b^4 = 2a^2$. In the first case, $\dfrac{a^2}{|b|} = \sqrt{2}$, and in the second case $\dfrac{b^2}{|a|} = \sqrt{2}$. In either case it is clear that a and b cannot both be rational, for it would make $\sqrt{2}$ rational.

8. A global maximum.

The maximum value is $f\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right) = \dfrac{1}{\sqrt{2}}$. For each fixed $x > 0$,

$$\lim_{y\to 0} \frac{y}{x^2+y^2} = 0 = \lim_{y\to\infty} \frac{y}{x^2+y^2},$$

so for each x, the quantity $y/(x^2+y^2)$ has a maximum value on $0 < y < \infty$, and at each such maximum point, $D_y(y/(x^2+y^2)) = 0$. But

$$D_y\left(\frac{y}{x^2+y^2}\right) = \frac{x^2-y^2}{(x^2+y^2)^2} = 0$$

if and only if $y = x$. With $y = x$, $f(x,y) = f(x,x) = \min\{x, 1/2x\}$, which is a maximum when $x = 1/2x$; i.e., when $x = 1/\sqrt{2}$. Thus the maximum value of the function is

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}.$$

SOLUTION II

We will use the fact that $x^2 + y^2 \geq 2xy$ for all $x > 0, y > 0$. If $x \leq y/(x^2+y^2)$, then $x \leq y/(x^2+y^2) \leq y/(2xy) = 1/(2x)$, which implies $x \leq 1/\sqrt{2}$, and equality holds iff $x = y = 1/\sqrt{2}$. If $x > 1/\sqrt{2}$ then $y/(x^2+y^2) \leq y/(2xy) = 1/(2x) < 1/\sqrt{2}$. Hence at least one of x and $y/(x^2+y^2)$ is $\leq 1/\sqrt{2}$. Consequently, $f(x,y) \leq 1/\sqrt{2}$ for all x, y, and $f\left(1/\sqrt{2}, 1/\sqrt{2}\right) = 1/\sqrt{2}$, so this is the maximum value.

9. A set containing 2014, 0 and -2.

From the fact that -1 is an integral root of the polynomial $2014 + 2014x$ with coefficients in S we know that $-1 \in S$. It follows that $1 \in S$ because it is a root of the polynomial $2014 - x - x^2 - \cdots - x^{2014}$. It now follows that $-2 \in S$ because it is a root of the polynomial $x^{11} - x^5 - x + 2014$.

10. An inequality.

We will prove the equivalent inequality obtained by subtracting 1 from each side of the desired one:

$$\frac{1+a+a^{n-1}+a^n}{a^2+a^3+\cdots+a^{n-2}} \geq \frac{4}{n-3}. \tag{1}$$

We first note that for $2 \leq k \leq n-2$,

$$1+a^n \geq a^k+a^{n-k}, \tag{2}$$

for this is equivalent to $(1-a^k)(1-a^{n-k}) \geq 0$, which is clear because both k and $n-k$ are positive integers, so both factors have the same sign. Similarly, for $2 \leq k \leq n-2$,

$$a+a^{n-1} \geq a^k+a^{n-k}, \tag{3}$$

for this is equivalent to $a(1-a^{k-1})(1-a^{n-k-1}) \geq 0$. Adding inequalities (2) for $2 \leq k \leq n-2$ we obtain

$$(n-3)(1+a^n) \geq 2(a^2+a^3+\cdots+a^{n-2}), \tag{4}$$

and adding inequalities (3) for $2 \leq k \leq n-2$ we have

$$(n-3)(a+a^{n-1}) \geq 2(a^2+a^3+\cdots+a^{n-2}). \tag{5}$$

Now add (4) and (5) to obtain

$$(n-3)(1+a+a^{n-1}+a^n) \geq 4(a^2+a^3+\cdots+a^{n-2}),$$

which is obviously equivalent to (1).