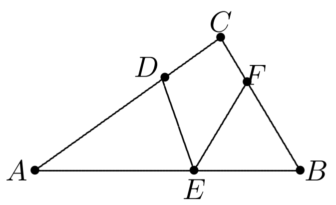
1. Measure of an angle.



$\boxed{\angle DEF = 43^\circ}$. We have

$$\begin{aligned}\angle DEF &= 180^\circ - \angle DEA - \angle FEB \\ &= 180^\circ - \frac{1}{2}(180^\circ - \angle DAE) - \frac{1}{2}(180^\circ - \angle FBE) \\ &= \frac{1}{2}(\angle DAE + \angle FBE) = \frac{1}{2}(180^\circ - \angle C) \\ &= \frac{1}{2}(180^\circ - 94^\circ) = 43^\circ.\end{aligned}$$

2. The 2013th term.

The 2013th term is $\boxed{3020^2 - 1}$. The integer $k^2 - 1 = (k-1)(k+1)$ is divisible by 3 if and only if k is not a multiple of 3. Thus the sequence begins $2^2 - 1,\ 4^2 - 1,\ 5^2 - 1,\ 7^2 - 1,\ 8^2 - 1, \cdots$, and the $2k$-th term is $a_{2k} = (3k+1)^2 - 1$, while $a_{2k+1} = (3k+2)^2 - 1$. Then

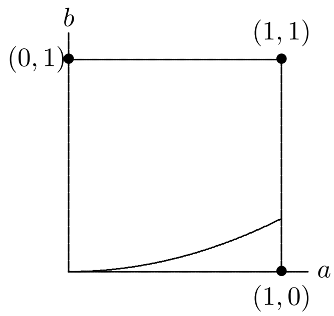
$$a_{2013} = a_{2\cdot 1006+1} = (3 \cdot 1006 + 2)^2 - 1 = 3020^2 - 1.$$

3 Probability of real roots.

The probability that the roots are real is $\boxed{1/12}$. The condition that $x^2 + ax + b = 0$ has real roots is that $a^2 - 4b \geq 0$; i.e., $b \leq a^2/4$. The area of this part of the square is

$$\int_0^1 \frac{a^2}{4} da = \frac{a^3}{12}\bigg|_0^1 = \frac{1}{12},$$

and the area of the whole square is 1, so with uniform distribution the probabililty in question is the ratio of these areas, namely 1/12.



4. Sum is 2013.

Their are eight such pairs. The sum of the integers from $n+1$ through $n+k$ inclusive is $k(2n+k+1)/2$, so we need $k(2n+k+1) = 4026 = 2 \cdot 3 \cdot 11 \cdot 61$. Clearly k must be smaller than $2n+k+1$, so the candidates for k are 1, 2, 3, 6, 11, 22, 33, 61. With $k = 1$ we have $2n + 2 = 4026$, giving $n = 2012$, so we have the pair $(2012, 1)$. With $k = 2$ we have $2n + 3 = 2013$, and $n = 1005$. With $k = 3$ we get $n = 669$; $k = 6$ gives $n = 332$; $k = 11$ gives $n = 177$; $k = 22$, $n = 80$; $k = 33$, $n = 44$; and with $k = 61$, $n = 2$.

5. 20th derivative at 0.

We'll show that $\boxed{f^{(20)}(0) = -2^3 \cdot 3^{18} \cdot 43}$. We use the Maclaurin series

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)x^k}{k!} \qquad \text{together with the known series} \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots.$$

Thus

$$\cos 3x = 1 - \frac{(3x)^2}{2!} + \frac{(3x)^4}{4!} - \cdots,$$

and $f^{(20)}(0)/20!$ is the coefficient of x^{20} in the series for $(x^2 + 4x + 4)\cos 3x$. The term in x^{20} will be

$$-\frac{x^2(3x)^{18}}{18!} + \frac{4(3x)^{20}}{20!},$$

and the coefficient of x^{20} is

$$-\frac{3^{18}}{18!} + \frac{4 \cdot 3^{20}}{20!} = \frac{-3^{18} \cdot 19 \cdot 20 + 4 \cdot 9 \cdot 3^{18}}{20!} = \frac{f^{(20)}(0)}{20!}.$$

Thus $f^{(20)}(0) = 4 \cdot 3^{18}(-19 \cdot 5 + 9) = -4 \cdot 3^{18} \cdot 86 = -2^3 \cdot 3^{18} \cdot 43$.

6. Predicted time remaining is one hour.

$\boxed{\text{It is possible}}$. (a) The distance travelled during the four hours in question is $\boxed{1600/33 \text{ miles}}$.
(b) The speed one hour after departure was $\boxed{25 \text{ mph}}$. Let $S(t)$ be the number of miles travelled during the first t hours of the journey. At time t, the average speed on the journey so far is $S(t)/t$, and the distance remaining is $100 - S(t)$, so the predicted remaining time remains constant at 1 hour provided that

$$\frac{100 - S(t)}{1} = \frac{S(t)}{t}. \tag{1}$$

This holds if and only if $100t - tS(t) = S(t)$; i.e.,

$$S(t) = \frac{100t}{t+1} = 100\left(1 - \frac{1}{t+1}\right).$$

This is a monotone increasing function of t, indicating that the auto is always moving forward, and satisfies condition (1), that the predicted remaining time will be one hour.

(a) During the four hour period in question the distance travelled is

$$S(9/2) - S(1/2) = \frac{450}{11/2} - \frac{50}{3/2} = \frac{900}{11} - \frac{100}{3} = \frac{1600}{33}$$

miles.

(b) $S'(t) = 100/(1+t)^2$, and at $t = 1$ the speed was $S'(1) = 25$ mph.

7. A periodic sequence.

Note first that if $m > n$ and $a_m = a_n$, then $a_{m+1} = a_{n+1}$, and the sequence beginning with this a_n will be periodic . Therefore it suffices to show that some value occurs more than once in the sequence. We will show that there are infinitely many terms of the sequence smaller than or equal to 2013. Consequently some value less than or equal to 2013 occurs more than once, and the sequence is periodic from this point on. Suppose that a_k is a term with $a_k > 2013$. If a_k is even then $a_{k+1} = a_k/2 < a_k$. If a_k is odd, then $a_{k+1} = a_k + 2013 < 2a_k$, so $a_{k+2} = a_{k+1}/2 < a_k$. Thus if $a_k > 2013$, one of the next two terms is less than a_k, and after a finite number of terms we obtain a term $a_{k+r} \le 2013$, and by the argument above, the sequence is eventually periodic.

8. A commutative operation.

Let $x, y \in S$, and let $z = xy$. We need to show that $yx = z$. We have

$$zy = (xy)y = x, \qquad \text{by (A2).} \tag{1}$$

Also,

$$y = z(zy) = zx \quad \text{by (A1) and (1).} \tag{2}$$

Then

$$yx = (zx)x = z, \quad \text{by (2) and (A2),}$$

so we have $yx = xy$, as desired.

9. Sum of interval lengths.

The sum of the lengths of the intervals is $\boxed{6}$. The graph of f consists of four branches with vertical asymptotes at $x = -1$, $x = -2$ and $x = -3$. In the interval $(-1, \infty)$, $f(x)$ is continuous and decreases monotonically from ∞ to 0, so $f(x) = 1$ at exactly one point r_1. In the interval $(-2, -1)$, $f(x)$ is continuous and decreases from ∞ to $-\infty$, and takes the value 1 at exactly one point r_2. In the interval $(-3, -2)$ $f(x)$ again takes the value 1 at exactly one point, r_3. In the interval $(-\infty, -3)$ $f(x)$ is negative. Thus, the set of points where $f(x) \ge 1$ is $(-3, r_3) \cup (-2, r_2) \cup (-1, r_1)$, which has total length $(r_3+3)+(r_2+2)+(r_1+1) = (r_1 + r_2 + r_3) + 6$, where r_1, r_2, r_3 are the roots of the equation $f(x) = 1$. Multiply this equation by $(x + 1)(x + 2)(x + 3)$ to get

$$(x+1)(x+2)(x+3)\left(\frac{1}{x+1} + \frac{2}{x+2} + \frac{3}{x+3}\right) = (x+1)(x+2)(x+3).$$

This is a polynomial equation of degree 3 having roots r_1, r_2, r_3. The coefficient of x^3 is 1, and the coefficient of x^2 will be the sum of the roots, $r_1 + r_2 + r_3$. The coefficient of x^2 on the right is $1 + 2 + 3 = 6$. The left member is

$$(x+2)(x+3) + 2(x+1)(x+3) + 3(x+1)(x+2),$$

where the coefficient of x^2 is again $1 + 2 + 3 = 6$. Thus when we transpose all terms to one side, the coefficient of x^2 is 0, so $r_1 + r_2 + r_3 = 0$, and the sum of the lengths of the intervals where $f(x) \ge 1$ is 6.

10. Sum of cubes greater than 40.

Suppose that $a_1+a_2+\cdots+a_n = 10$ and $a_1^2+a_2^2+\cdots+a_n^2 > 20$. Then by the Cauchy-Schwartz inequality,

$$\begin{aligned}
10(a_1^3 + a_2^3 + \cdots + a_n^3) &= (a_1 + a_2 + \cdots + a_n)(a_1^3 + a_2^3 + \cdots + a_n^3) \\
&\geq \left(\sqrt{a_1}\sqrt{a_1^3} + \sqrt{a_2}\sqrt{a_2^3} + \cdots + \sqrt{a_n}\sqrt{a_n^3}\right)^2 \\
&= (a_1^2 + a_2^2 + \cdots + a_n^2)^2 \\
&> 400.
\end{aligned}$$

Thus $a_1^3 + a_2^3 + \cdots + a_n^3 > 40$.