Each problem number is followed by an 11-tuple $(a_{10}, a_9, a_8, a_7, a_6, a_5, a_4, a_3, a_2, a_1, a_0)$, where a_k is the number of teams that scored k points on the problem.

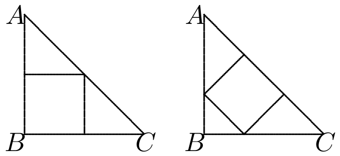
1. Units digit of S_{2011}. (28,1,1,0,1,0,9,2,5,5,24)

The units digit of S_{2011} is $\boxed{1}$. We show this by observing that $S_1 = 1$ has units digit 1, and that S_{n+10} always has the same units digit as does S_n. Modulo 10, the sequence of squares of positive odd integers begins (1,9,5,9,1), after which this cycle repeats itself, because $(n+10)^2 = n^2 + 20n + 100 \equiv n^2 \pmod{10}$. Then modulo 10, $S_{n+10} - S_n$ will always be the sum of 10 consecutive terms in this sequence, which will be

$$1 + 9 + 5 + 9 + 1 + 1 + 9 + 5 + 9 + 1 \equiv 0,$$

wherever in the sequence one begins. Thus $S_{n+10} \equiv S_n \pmod{10}$.

2. Area of a square. (60,3,0,2,0,2,1,0,1,0,7)



The area is $\boxed{\frac{8}{9}S}$. The legs of the right triangle have length $2\sqrt{S}$, so the hypotenuse is $2\sqrt{2S}$. Then the second inscribed square has side $\frac{2\sqrt{2S}}{3}$, because the smaller triangles at A and C are also isosceles right triangles. Thus the area of the second inscribed square is $\left(\frac{2\sqrt{2S}}{3}\right)^2 = \frac{8S}{9}$.

3 Speshul integers. (31,0,0,2,0,0,4,7,2,0,30)

Well, of course, $\boxed{\text{every positive integer}}$ is speshul. Here are two different ways to get the m and n we need.

(1) To make $mn + 1 = km + kn$, start by taking $m = k + 1$. Then we want $kn + n + 1 = km + kn$; i.e., $n + 1 = km$, so $n = km - 1$ does it. Thus: $m = k + 1$, $n = km - 1 = k^2 + k - 1$.

(2) Take $m = 2k - 1$, $n = 2k + 1$. Then

$$\frac{mn + 1}{m + n} = \frac{4k^2}{4k} = k.$$

4. Divisor and remainder. (20,0,2,1,1,1,2,0,0,9,40)

The unique such pair is $\boxed{(d, r) = (71, 52)}$. From $904 = ad + r$, $1259 = bd + r$ and $2040 = cd + r$ we obtain on subtraction,

$$355 = (b - a)d \quad \text{and} \quad 781 = (c - b)d.$$

Now, $781 = (2)(355) + 71$, and both 781 and 355 are multiples of d, so 71 is a multiple of d. But 71 is prime, so d can only be 71 (the condition $0 < r < d$ requires $d > 1$), and then $904 = (12)(71) + 52$ shows that r must be 52. One checks that $1259 = (17)(71) + 52$ and $2040 = (28)(71) + 52$, so $(d, r) = (71, 52)$ is the unique solution.

5. Bigger than 2011^2?. (18,0,1,0,2,0,2,4,5,6,38)

Such numbers $\boxed{\text{do exist}}$. Let $f(x) = x^2 + \sqrt{2011 - x}$ for $0 \le x \le 2011$. It suffices to show that there exists $x \in [0, 2011]$ with $f(x) > 2011^2$. We note that $f(2011) = 2011^2$, so it suffices to show that $f'(x)$ is negative throughout an interval $(2011 - \epsilon, 2011)$ for some $\epsilon > 0$. Well, $f'(x) = 2x - 1/(2\sqrt{2011 - x})$, and for $0 \le x \le 2011$, $2x < 4022$. But $1/(2\sqrt{2011 - x}) > 4022$ when $1/\sqrt{2011 - x} > 8044$; i.e., when $2011 - x < 1/8044^2$. Thus, when

$$2011 - \frac{1}{8044^2} < x < 2011,$$

$f'(x) < 0$, so $f(x)$ decreases in this interval to the value 2011^2 at $x = 2011$, and hence $f(x) > 2011^2$ throughout this interval.

6. From 11 to 2011. (38,1,1,1,0,0,1,2,0,0,32)

Yes, it can be done in many ways (infinitely many, in fact). A key observation is that with any $n > 1$, $n = (n - 1) + 1$ can be replaced by $(n - 1) \cdot 1 = n - 1$, so if we obtain any integer larger than 2011, we can step down from there to 2011 in steps of 1. Thus, one solution is

$$11 = 6 + 5 \to 30 = 15 + 15 \to 225 = 10 + 215 \to 2150 = 2149 + 1 \to 2149,$$

etc., down to 2011.

7. Not a perfect square. (11,0,0,0,0,1,0,2,2,1,59)

We show that $(n + 5)^2 < a_n < (n + 6)^2$ for all n, and the claim follows. Using the facts that $4 < \sqrt{19} < 5$ and $9 < \sqrt{99} < 10$ we have

$$\begin{aligned} a_n &= \lfloor n^2 + 2\sqrt{19}n + 19 + 2n + \sqrt{99} \rfloor \\ &\le \lfloor n^2 + 12n + 29 \rfloor \\ &= \lfloor (n + 6)^2 - 7 \rfloor \\ &< (n + 6)^2, \end{aligned}$$

and

$$\begin{aligned} a_n &\ge \lfloor n^2 + 10n + 28 \rfloor \\ &= \lfloor (n + 5)^2 + 3 \rfloor \\ &> (n + 5)^2. \end{aligned}$$

8. $P(11) = 2011$**.** (35,1,6,0,1,0,0,0,0,0,33)

We show that $\boxed{P(6) = 441}$. From the fact that $2011 = P(11) \geq a_n \cdot 11^n \geq 11^n$ we know that $n \leq 3$ because $11^3 = 1331 < 2011 < 11^4$. Also, $2011 < 2 \cdot 11^3$, so $a_3 \leq 1$, and thus if $n = 3$ then $a_3 = 1$. We show that $n = 3$. If $n \leq 2$, then $P(x) = a_0 + a_1 x + a_2 x^2 \leq 10 + 10x + 10x^2$, and $2011 = P(11) \leq 10 + 110 + 1210 < 2011$, a contradiction. Thus $n = 3$, $a_3 = 1$, and $P(x) = a_0 + a_1 x + a_2 x^2 + x^3$. Then $2011 = P(11) = a_0 + a_1 \cdot 11 + a_2 \cdot 11^2 + 1331$, whence $a_0 + a_1 \cdot 11 + a_2 \cdot 11^2 = 2011 - 1331 = 680$.

Because $0 \leq a_0 + a_1 \cdot 11 \leq 10 + 110 = 120$, we have $680 \geq a_2 \cdot 11^2 \geq 680 - 120 = 560$, and therefore $6 > 680/121 \geq a_2 \geq 560/121 > 4$. Hence $a_2 = 5$, and

$$2011 = a_0 + a_1 \cdot 11 + 5 \cdot 121 + 1331 = a_0 + 11a_1 + 1936,$$

so $a_0 + 11a_1 = 75$. The constraints on a_0 and a_1 then determine that $a_1 = 6$ and $a_0 = 9$. Thus $P(x) = 9 + 6x + 5x^2 + x^3$. Note that this also satisfies $P(1) = 21$ (which was given but not used for the solution). Finally, then, $P(6) = 9 + 36 + 180 + 216 = 441$.

9. Fractional part of $\sqrt{n}$. (8,0,2,0,0,0,0,0,0,0,66)

Let r be the integer such that $r - 1 < \sqrt{n} < r$. If $r = 2$, then $n = 2$ or 3 and one can easily verify directly that the required inequality holds. So we assume that $r \geq 3$, in which case $(r-1)^2 \geq r$, and hence $n > r$. Let $k = r^2 - n$. Then $k > 0$ so $k \geq 1$, and we have

$$x_n = \sqrt{n} - \lfloor \sqrt{n} \rfloor = \sqrt{n} - (r - 1),$$

and then

$$\begin{aligned} x_n + \frac{1}{2n} &= \sqrt{n} - r + 1 + \frac{1}{2n} \\ &= \sqrt{r^2 - k} - r + 1 + \frac{1}{2n} \\ &\leq \sqrt{r^2 - 1} - r + 1 + \frac{1}{2n} \\ &\leq \sqrt{r^2 - 1} - r + 1 + \frac{1}{2r} \\ &= 1 - \left(r - \sqrt{r^2 - 1} - \frac{1}{2r} \right) \\ &= 1 - \left(\frac{1}{r + \sqrt{r^2 - 1}} - \frac{1}{2r} \right) < 1, \end{aligned}$$

q.e.d.

10. Numerator divisible by 2011. (3,1,0,0,0,0,0,0,0,0,72)

We rewrite

$$\begin{aligned}\frac{p}{q} &= \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{1339}+\frac{1}{1340}\right)-2\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{1340}\right)\\ &= \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{1339}+\frac{1}{1340}\right)-\left(1+\frac{1}{2}+\cdots+\frac{1}{670}\right)\\ &= \frac{1}{671}+\frac{1}{672}+\cdots+\frac{1}{1340},\end{aligned}$$

Now group these terms in pairs as follows:

$$\begin{aligned}\frac{p}{q} &= \left(\frac{1}{671}+\frac{1}{1340}\right)+\left(\frac{1}{672}+\frac{1}{1339}\right)+\cdots+\left(\frac{1}{1005}+\frac{1}{1006}\right)\\ &= \frac{1340+671}{671\cdot 1340}+\frac{1339+672}{672\cdot 1339}+\cdots+\frac{1006+1005}{1005\cdot 1006}\\ &= 2011\left(\frac{1}{671\cdot 1340}+\frac{1}{672\cdot 1339}+\cdots+\frac{1}{1005\cdot 1006}\right)\\ &= 2011\frac{r}{s},\end{aligned}$$

for some integers r and s, where all prime factors of s are smaller than 2011. We now have $2011rq = ps$, 2011 is prime, and 2011 does not divide s. Therefore $2011|p$, q.e.d.