Each problem number is followed by an 11-tuple $(a_{10}, a_9, a_8, a_7, a_6, a_5, a_4, a_3, a_2, a_1, a_0)$, where a_k is the number of teams that scored k points on the problem.

1. $\lfloor \log_3 n \rfloor$ **is a multiple of 3.** (28,7,13,3,5,2,0,1,2,3,9,12)

The answer is $\boxed{1338/2010 = 223/335}$. From the definitions of greatest integer function and of logarithm, we have

$$\lfloor \log_3 n \rfloor = 3k \iff 3k \le \log_3 n < 3k+1 \iff 3^{3k} \le n < 3^{3k+1}.$$

For $k = 0$ we get the integers $\{1, 2\}$, and for $k = 1$ we get $\{n : 27 \le n < 81\}$. With $k = 2$ and the restriction that n be no greater than 2010, we obtain $\{n : 729 \le n \le 2010\}$. Thus there are $2 + (80 - 26) + (2010 - 728) = 1338$ integers satisfying the condition, giving a probability of 1338/2010=223/335.

2. Find vertices given midpoints. (55,0,0,0,2,1,1,0,1,1,24)

The vertices are $\boxed{A = (-7, -10),\ B = (9, -4),\ C = (7, 12),\ D = (-1, 8),\ E = (3, 0)}$. Regarding points as vectors in the plane, we have

$$
\begin{aligned}
A + B & & & & &= 2P \\
& B + C & & & &= 2Q \\
& & C + D & & &= 2R \\
& & & D + E &&= 2S \\
A & & & & + E &= 2T.
\end{aligned}
$$

The augmented matrix of this system is

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 2P \\
0 & 1 & 1 & 0 & 0 & 2Q \\
0 & 0 & 1 & 1 & 0 & 2R \\
0 & 0 & 0 & 1 & 1 & 2S \\
1 & 0 & 0 & 0 & 1 & 2T
\end{pmatrix}
$$

Using standard Gaussian elimination on this system we obtain the triangularized matrix

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 2P \\
0 & 1 & 1 & 0 & 0 & 2Q \\
0 & 0 & 1 & 1 & 0 & 2R \\
0 & 0 & 0 & 1 & 1 & 2S \\
0 & 0 & 0 & 0 & 2 & 2S - 2R + 2Q - 2P + 2T
\end{pmatrix}
$$

Thus

$$
\begin{aligned}
E &= S - R + Q - P + T = (3, 0) \\
D &= 2S - E = (-1, 8) \\
C &= 2R - D = (7, 12) \\
B &= 2Q - C = (9, -4) \\
A &= 2P - B = (-7, -10).
\end{aligned}
$$

3 Real part of $1/(1 - z)$. (42,1,0,0,0,0,2,0,0,1,39)

The real part has constant value $\boxed{1/2}$. For $z = x + yi$ with x and y real, let

$$f(z) = \frac{1}{1 - z} = \frac{1}{1 - x - yi} = \frac{1 - x + yi}{(1 - x - yi)(1 - x + yi)} = \frac{1 - x + yi}{1 - 2x + x^2 + y^2}.$$

If $|z| = 1$ then $x^2 + y^2 = 1$, and the real part of $f(z)$ is $(1 - x)/(2 - 2x) = 1/2$.

4. Length of a segment. (53,1,0,1,2,1,0,2,7,5,13)

The length $\boxed{|BE| = \sqrt{3} - 1}$. Each side of the equilateral triangle has length $2\sqrt{3}$. Let O be the center of the circle, and F the point of tangency of side AB, and K the point of tangency on line DE. Then $OKEF$ is a square of side 1, so $|BE| = |BF| - 1 = \sqrt{3} - 1$.

5. Sum is 1/2010. (18,0,1,2,6,2,2,3,0,3,48)

Two of the several solutions are $\boxed{(m, n) = (1005, 2009) \text{ and } (1340, 4019)}$. We use the identity

$$\frac{1}{k(k + 1)} = \frac{1}{k} - \frac{1}{k + 1}$$

to telescope the desired sum:

$$\sum_{k=m}^{n} \frac{1}{k(k + 1)} = \sum_{k=m}^{n} \left(\frac{1}{k} - \frac{1}{k + 1}\right) = \frac{1}{m} - \frac{1}{n + 1}.$$

Thus we seek integers m, n satisfying

$$\frac{1}{m} - \frac{1}{n + 1} = \frac{1}{2010}.$$

Clearing of fractions, collecting terms and rearranging yields the equivalent equation

$$mn + 2011m - 2010n = 2010,$$

and thus

$$(m - 2010)(n + 2011) = 2010 - 2010 \cdot 2011 = -2010^2,$$

so each factorization of -2010^2 into a product of two integers yields a solution for (m, n) in integers. The solutions given above come from $m - 2010 = -1005$ and -670, respectively. Any of the negative factors of 2010^2 from -1 to -1675 are suitable choices for $m - 2010$.

6. Both have value 7. (28,0,1,1,0,1,0,1,2,0,51)

It is well-known that f is continuous, and we want to show that there is a number r such that $f(r) + f(7-r) = 7$. Let $g(x) = f(x) + f(7-x)$. Then g is continuous and $g(2) = f(2) + f(5) < 7 < f(3) + f(4) = g(3)$, so by the Intermediate Value Theorem there is a number r between 2 and 3 such that $g(r) = 7$; i.e., $f(r) + f(7-r) = 7$.

7. Limit of a sequence. (4,0,1,0,0,2,5,18,4,3,48)

The limit is $\boxed{\frac{\pi}{2}(\sin 2 - \sin 1)}$. Integrate by parts using

$$u = \text{Arctan } nx, \;\; du = \frac{n}{1 + n^2 x^2} dx, \;\; dv = \cos x \, dx, \;\; v = \sin x$$

to get

$$\int_1^2 (\cos x)(\text{Arctan } nx) dx = (\text{Arctan } nx)(\sin x)\big|_1^2 - \int_1^2 (\sin x)\frac{n}{1 + n^2 x^2} dx$$

$$= (\text{Arctan } 2n)(\sin 2) - (\text{Arctan } n)(\sin 1) - \int_1^2 (\sin x)\frac{n}{1 + n^2 x^2} dx.$$

Now

$$\left| \int_1^2 (\sin x)\frac{n}{1 + n^2 x^2}) dx \right| \leq \int_1^2 |\sin x|\frac{n}{1 + n^2 x^2} dx \leq \int_1^2 \frac{n}{1 + n^2 x^2} dx$$

$$\leq \int_1^2 \frac{1}{n} dx = \frac{1}{n} \to 0 \text{ as } n \to \infty.$$

Also,

$$\lim_{n \to \infty} \text{Arctan } 2n = \lim_{n \to \infty} \text{Arctan } n = \frac{\pi}{2},$$

so the desired limit is

$$\frac{\pi}{2}(\sin 2 - \sin 1).$$

8. An unbounded sequence. (19,0,1,1,2,4,2,2,2,3,49)

If $a_0 = 7$ the sequence begins 7, 44, 22, 11, 116, 58, 29, and one may notice that every third term is odd. We show by induction that a_{3n} is odd for all n, and that $a_{3n+3} > a_{3n}$ for all n. As all terms are evidently integers, this will show that the sequence is unbounded.

That a_0 is odd is given. Suppose that $a_{3n} = 2k + 1$, with $k \geq 3$. Then $a_{3n+1} = 4k^2 + 4k - 4$, $a_{3n+2} = 2k^2 + 2k - 2$ and $a_{3n+3} = k^2 + k - 1$, which is odd because $k(k + 1)$ is even. It remains to show that $k^2 + k - 1 > 2k + 1$, and this is equivalent to $k(k - 1) > 2$, which is true for all $k \geq 3$.

9. Divisible by 2010. (16,0,3,0,0,1,0,0,1,1,63)

We use the fact that $x^n - y^n$ is always divisible by $x - y$. Observe that $2013 - 1827 = 1678 - 1492 = 186$, from which it follows that $(2013^n - 1827^n) - (1678^n - 1492^n)$ is divisible by $186 = 31 \cdot 6$, and therefore by 6. Also, $2013 - 1678 = 1827 - 1492 = 335$, whence $(2013^n - 1678^n) - (1827^n - 1492^n)$ is divisible by 335. As 6 and 335 are relatively prime, it follows that $1492^n - 1678^n - 1827^n + 2013^n$ is divisible by $(6)(335) = 2010$.

SECOND SOLUTION

The prime factorization of 2010 is $2 \cdot 3 \cdot 5 \cdot 67$. Modulo 2, $1492^n - 1678^n - 1827^n + 2013^n \equiv 0 - 0 - 1 + 1 = 0$. Mod 3 we have $1^n - 1^n - 0^n + 0^n = 0$. Mod 5 we have $2^n - 3^n - 2^n + 3^n = 0$, and mod 67 we have $18^n - 3^n - 18^n + 3^n = 0$. It follows that $1492^n - 1678^n - 1827^n + 2012^n$ is divisible by $2 \cdot 3 \cdot 5 \cdot 67 = 2010$.

10. Three real roots. (7,0,0,0,0,0,0,1,0,3,74)

Because $c/a < 0$, $ax^2 + c = 0$ has two real roots, $x_1 = -\sqrt{-c/a} < 0$ and $x_2 = \sqrt{-c/a} > 0$. Then $P(x_1) = x_1(x_1^2 + b) = x_1(b - c/a) > 0$, and $P(x_2) = x_2(b - c/a) < 0$. Thus $P(x)$ has zeros in $(-\infty, x_1), (x_1, x_2)$ and (x_2, ∞).

SECOND SOLUTION

Note that $P(0) = c$ and $P(-a) = c - ab$. If $a > 0$, then $c < 0$, and $ab < c < 0$ so $c - ab > 0$. Thus $-a < 0$ while $P(-a) > 0$ and $P(0) = c < 0$, so there are roots in $(-\infty, -a), (-a, 0)$ and $(0, \infty)$.

If $a < 0$, then $ab > c > 0$ and $-a > 0$. Thus we have $P(-a) = c - ab < 0$ while $-a > 0$ and $p(0) = c > 0$, so there are roots in $(-\infty, 0), (0, -a)$ and $(-a, \infty)$.