TWENTY-THIRD ANNUAL
NORTH CENTRAL SECTION MAA
TEAM CONTEST

November 16, 2019, 9:00 a.m. to 12:00 noon

NO BOOKS, NOTES, CALCULATORS, OR NON-TEAM-MEMBERS may be consulted. NO CELL PHONES OR OTHER ELECTRONIC DEVICES.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team name and problem number should be given at the top of each sheet of paper submitted.

Each problem counts 10 points. Partial credit is given for significant but incomplete work. For full credit, answers must be fully justified. Have fun!

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1. Divisible by 19.

Given that x and y are integers, prove that $2x + 6y$ is divisible by 19 if and only if $5x - 4y$ is divisible by 19.

2. Intersections of quadratics .

Isn't it true that the graphs of two quadratic equations can intersect in at most four points? Explain, then, how it is possible for the five points $(-1, 6)$, $(3, -2)$, $(1, 1)$, $(0, 4)$, and $(2, 0)$ to satisfy both of the following quadratic equations:

$$2x^2 - y^2 - xy - 4x + 4y = 0 \tag{1}$$

$$6x^2 - y^2 + xy - 16x + 2y + 8 = 0. \tag{2}$$

3. No solutions in integers.

Show that there are no positive integers m and n such that $m(m + 1) = n(n + 2)$

4. Square populations.

The population of a certain geographical entity in 1990 was a perfect square. In 2000 the population had grown by 1000 and was 1 more than a perfect square. In 2010 the population had again grown by 1000, and was again a perfect square. Find all possible values of the 1990 population (and show that no others are possible).

5. An integer-valued function.

Let $f(x) = \frac{1}{5}x^5 + \frac{1}{3}x^3 + \frac{7}{15}x$. Show that whenever x is an integer, so is $f(x)$.

6. A convergent series.

Show that the series

$$\sum_{k=1}^{\infty} \ln\left(1 + \frac{1}{k^2 + 2k}\right)$$

converges, and find its sum.

7. Multiplying two arithmetic sequences.

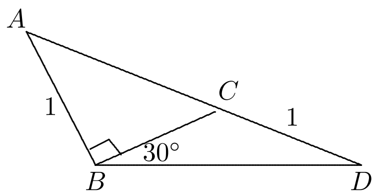
A sequence is obtained by multiplying the corresponding terms of two arithmetic sequences. For example, if the two given sequences begin 3, 5, 7,. . . and 4, 9, 14,. . ., then the product sequence begins 12, 45, 98, . . . Given that the first three terms of the product sequence are 1216, 1360, and 1384, find the seventh term.

8. Probability that it is divisible by 11.

Among all nine-digit integers in which each of the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 occurs exactly once, one is chosen at random. Find the probability that it is divisible by 11. Express your answer as a rational fraction in lowest teems.

9. Length of a segment.

In the triangle ABD, side AB has length 1. Point C lies on side AD with CB perpendicular to AB, and the length of CD is 1. Angle CBD is 30°. What is the length of segment AC?



10. 2019 sums of consecutive integers.

The number 9 can be expressed in exactly two different ways as the sum of 2 or more consecutive positive integers, namely as $4+5$ and as $2+3+4$. What is the smallest positive integer that can be expressed in exactly 2019 different ways as the sum of two or more consecutive positive integers?