TWENTY-SECOND ANNUAL
NORTH CENTRAL SECTION MAA
TEAM CONTEST

November 10, 2018, 9:00 a.m. to 12:00 noon

NO BOOKS, NOTES, CALCULATORS, OR NON-TEAM-MEMBERS may be consulted. NO CELL PHONES OR OTHER ELECTRONIC DEVICES.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team name and problem number should be given at the top of each sheet of paper submitted.

Each problem counts 10 points. Partial credit is given for significant but incomplete work. For full credit, answers must be fully justified. Have fun!

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1. Multiples of 23.

Prove that $3^n 2^{3n} - 1$ is divisible by 23 for every positive integer n.

2. The 2018th term.

Let $\{x_n\}$ and $\{y_n\}$ be sequences of real numbers satisfying the recursions $x_{n+1} = x_n^3 - 3x_n$ and $y_{n+1} = y_n^3 - 3y_n$. If $x_0^2 - y_0 = 2$ find (with proof) the value of $x_{2018}^2 - y_{2018}$.

3. Sums of fourth powers.

Note that

$$\begin{aligned} 1^4 + 2^4 + 3^4 &= 2 \cdot 7^2 \\ 1^4 + 3^4 + 4^4 &= 2 \cdot 13^2 \\ 2^4 + 3^4 + 5^4 &= 2 \cdot 19^2 \\ 1^4 + 4^4 + 5^4 &= 2 \cdot 21^2. \end{aligned}$$

Find, and prove, a general identity of which these are special cases.

4. Each face occurs twice.

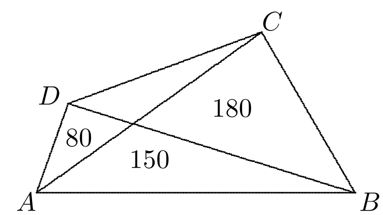
A fair six-sided die is rolled 12 times. What is the probability that each face occurs exactly twice? Express your answer as a rational fraction with numerator and denominator each a product of prime powers.

5. A subset of the first 2018 positive integers.

Let S be the set of integers from 1 to 2018. A certain subset B of S has 1025 elements. Prove that there are two elements of B which differ by 31.

6. Area of a quadrilateral.

The quadrilateral $ABCD$ is partitioned into four triangles by means of the diagonals AC and BD. The areas of three of the triangles are indicated in the diagram. What is the area of the quadrilateral?



7. A quadratic equation with integral roots.

Find all rational numbers r such that the solutions of the quadratic equation

$$rx^2 + (r+1)x + r = 1$$

are integers.

8. $2018 + xy$ is a perfect square.

Determine whether there exist four distinct positive integers such that adding the product of any two of them to 2018 yields a perfect square.

9. A max/min problem.

For real numbers x, y, z, define

$$f(x,y,z) = \min\{x - y^2, y - z^2, z - x^2\}.$$

Thus, for example, $f(1,2,3) = \min\{-3, -7, 2\} = -7$. Find the maximum value of $f(x,y,z)$.

10. A logarithmic inequality.

Suppose that a, b, c, d are positive numbers such that $a + b = c + d$. Prove that

$$a \ln a + b \ln b \geq a \ln c + b \ln d.$$