TWENTY-FIRST ANNUAL
NORTH CENTRAL SECTION MAA
TEAM CONTEST

November 11, 2017, 9:00 a.m. to 12:00 noon

NO BOOKS, NOTES, CALCULATORS, OR NON-TEAM-MEMBERS may be consulted. NO CELL PHONES OR OTHER ELECTRONIC DEVICES.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team identification and problem number should be given at the top of each sheet of paper submitted.

Each problem counts 10 points. Partial credit is given for significant but incomplete work. For full credit, answers must be fully justified. Have fun!

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1. An averaging operation.

Define the operation * on the real numbers by $a * b = (a + b)/2$. Find all numbers m such that $m * (m * 2017) = m$.

2. An inequality.

Show that for all integers $k \geq 0$,

$$1^{2k+1} + 2^{2k+1} + 3^{2k+1} \geq 6^{k+1}.$$

3. Probability the sum is odd.

An urn contains 9 balls numbered 1, 2, 3, 4, 5, 6, 7, 8, 9. If five of the balls are drawn at random, without replacement, from the urn, what is the probability that the sum of the numbers on the balls drawn is odd?

4. A sum of reciprocals.

Find all (unordered) pairs $\{x, y\}$ of integers satisfying $1/x + 1/y = 1/14$.

5. Floor-function integral.

Evaluate

$$\int_2^4 \lfloor x^3 - 6x^2 + 12x - 6 \rfloor \, dx,$$

where for every real number t, $\lfloor t \rfloor$ denotes the greatest integer less than or equal to t.

6. A sum of 2017 terms.

For positive integers n, let

$$a_n = \frac{2^n}{2^{2n+1} - 2^{n+1} - 2^n + 1}.$$

Show that $a_1 + a_2 + \cdots + a_{2017} < 1$.

7. Counting permutations.

Determine the number of permutations $(a_1, a_2, a_3, \ldots, a_{10})$ of $(1, 2, 3, \ldots, 10)$ which satisfy $a_1 > a_2 > a_3 > a_4 > a_5$ and $a_5 < a_6 < a_7 < a_8 < a_9 < a_{10}$.

8. A limit.

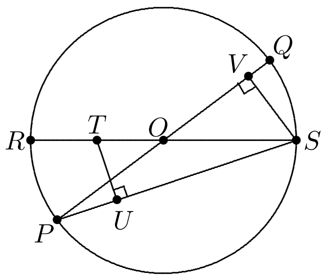
Evaluate

$$\lim_{x \to \infty} x\left(\sqrt{x^2+1} - \sqrt[3]{x^3+1}\right).$$

9. Equal products of side lengths.

In a circle centered at O, two diameters are PQ and RS. The midpoint of OR is T, U is the foot of the perpendicular from T to the chord PS, and V is the foot of the perpendicular from S to PQ. Prove that

$$(OT)(PV) = (PU)(PS).$$



10. Infinitely many terms divisible by 2017.

The sequence $\{a_n\}$ is defined recursively by $a_1 = 8$, $a_2 = 45$ and $a_{n+2} = a_{n+1}^2 - a_n$. Prove that there are infinitely many terms of the sequence divisible by 2017.