TWENTIETH ANNUAL
NORTH CENTRAL SECTION MAA
TEAM CONTEST

November 12, 2016,   9:00 a.m. to 12:00 noon

NO BOOKS, NOTES, CALCULATORS, OR NON-TEAM-MEMBERS may be consulted. NO CELL PHONES OR OTHER ELECTRONIC DEVICES.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team identification and problem number should be given at the top of each sheet of paper submitted.

Each problem counts 10 points. Partial credit is given for significant but incomplete work. For full credit, answers must be fully justified. Have fun!

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1. A geometric progression.
Find all real numbers x such that the numbers 4^{3x-1}, 8^{2x}, 16 are consecutive terms in a geometric progression.

2. A 2016 evaluation.
If x is a positive number such that $x^2 + 1/x^2 = 2016$, find the value of $x^3 + 1/x^3$.

3. Some cubic polynomials.
Find all polynomials of the form $P(x) = x^3 - ax^2 + bx - c$ which factor as $(x-a)(x-b)(x-c)$.

4. A sum of squared sines.
Find the value of the sum

$$S = \sin^2 0° + \sin^2 1° + \sin^2 2° + \cdots + \sin^2 90°.$$

5. Lots of powers of 2.

The definition of the ceiling function $\lceil \cdot \rceil$ is that $\lceil x \rceil$ is the least integer greater than or equal to x. Prove that for every positive integer n, $\lceil (1 + \sqrt{3})^{2n} \rceil$ is divisible by 2^{n+1}.

6. Sides of a triangle.

The altitudes of a triangle are 3, 4 and 6. Find the lengths of the sides.

7. Five triangles of equal area.

In the triangle ABC, $AB = 30$ and $AC = 32$. Points D and F lie on side AB in the order $AFDB$, and points E and G lie on AC in the order $AGEC$. The five triangles AGF, FGE, FED, DEC and DCB all have the same area. Find the length of FD.

8. Divisible by 2^{2016}.

Does there exist an integer of 2016 decimal digits, each digit of which is 6 or 7, which is divisible by 2^{2016}?

9. All terms integers?.

A sequence is defined recursively by $a_1 = 1$, $a_2 = 7$, and for $n \geq 3$, $a_n = \frac{a_{n-1}^2 + 13}{a_{n-2}}$. Determine whether all terms of the sequence are integers.

10. An inequality.

Prove that if a, b and c are positive real numbers, then

$$a^3 + b^3 + c^3 \geq (a^2)\sqrt{bc} + (b^2)\sqrt{ac} + (c^2)\sqrt{ab}.$$