NINETEENTH ANNUAL
NORTH CENTRAL SECTION MAA
TEAM CONTEST

November 14, 2015, 9:00 a.m. to 12:00 noon

NO BOOKS, NOTES, CALCULATORS, OR NON-TEAM-MEMBERS may be consulted. NO CELL PHONES OR OTHER ELECTRONIC DEVICES.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team identification and problem number should be given at the top of each sheet of paper submitted.

Each problem counts 10 points. Partial credit is given for significant but incomplete work. For full credit, answers must be fully justified. Have fun!

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1. Surface area of a box.

The length of an interior diagonal of a rectangular box is 8 cm., and the sum of the lengths of all the edges of the box is 36 cm. Find the surface area of the box.

2. Minimum value of a sequence.

In an arithmetic progression $\{t_1, t_2, t_3 \ldots\}$, the first term is $t_1 = 98$, and $t_{13} = 89$. Let $T_n = t_n + t_{n+1} + \cdots + t_{n+6}$. Find the minimum value of $|T_n|$.

3. Counting rational numbers.

Determine how many rational numbers there are between 0 and 1 having the property that when written as a fraction (quotient of integers) in lowest terms, the product of the numerator and the denominator is 17!.

4. Sets with sum 2015.

Find and list all sets of two or more consecutive positive integers with sum 2015.

5. Divisibility.

For what integers n is the integer $n^3 - 8n^2 + 2n$ divisible by $n^2 + 1$?

6. Limit of a sequence.

Find all positive real numbers c satisfying

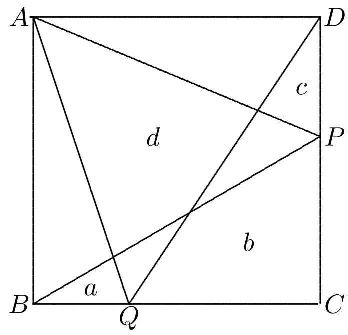
$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+kc} = \frac{1}{c}.$$

7. n^{2015} as a sum of squares.

Show that there are infinitely many pairs (a, b) of positive integers such that $a^2 + b^2 = n^{2015}$ for some integer n.

8. Equal areas.

In the figure at the right, $ABCD$ is a square, P and Q are any two points on the interiors of sides CD and BC, respectively. Lower case letters a, b, c and d denote the areas of the minimal segments of the figure in which they lie; triangles in the case of a and c; quadrilaterals in the case of b and d. Prove that $a + b + c = d$.



9. A bound on the zeros of a polynomial.

Let $P(x) = x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, where each a_k is a complex number and $|a_k| \leq 6$. Prove that if α is a complex number satisfying $P(\alpha) = 0$, then $|\alpha| < 7$.

10. Sum of squares less than 2015.

The real numbers $a_1, a_2, \ldots, a_{31}$ lie in the interval $[-5, 13]$, and $a_1 + a_2 + \cdots + a_{31} = 0$. Show that $a_1^2 + a_2^2 + \cdots + a_{31}^2 \leq 2015$.