EIGHTEENTH ANNUAL
NORTH CENTRAL SECTION MAA
TEAM CONTEST

November 15, 2014, 9:00 a.m. to 12:00 noon

NO BOOKS, NOTES, CALCULATORS, OR NON-TEAM-MEMBERS may be consulted. NO CELL PHONES OR OTHER ELECTRONIC DEVICES.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team identification and problem number should be given at the top of each sheet of paper submitted.

Each problem counts 10 points. Partial credit is given for significant but incomplete work. For full credit, answers must be fully justified. Have fun!

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1. Equation of a circle.

A circle in the (x, y)-plane passes through the origin and the point $A(1, 3)$. The point B diametrically opposite from A on the circle has abscissa 2 (i.e., its first coordinate is 2). Find the equation of the circle.

2. Simultaneous equations.

Find all real number pairs (x, y) satisfying the following equations:

$$x + y - \lfloor y \rfloor = 30.1$$
$$x + \lfloor x \rfloor + \lfloor y \rfloor = 71.7$$

(Recall that for real numbers u, $\lfloor u \rfloor$ denotes the greatest integer not exceeding u.)

3. 2014 concentric circles.

2014 concentric circles with radii 1, 2, 3, . . ., 2014 are drawn in the plane. The interior of the circle of radius 1 is colored black, the region between the circles of radius 1 and 2 is colored white, that between the circles of radius 2 and 3 is colored black, and so on, alternating black and white regions between circles of radius k and $k + 1$. What is the ratio of the total area of the black regions to the area of the circle of radius 2014? Express your answer as a rational fraction in lowest terms.

4. Cubic polynomial game.

Adolf and Bertha jointly construct a cubic polynomial as follows: Adolf chooses an integer $a \neq 0$, then Bertha chooses an integer b, then Adolf chooses an integer c and finally Bertha chooses an integer d, and they form the polynomial $f(x) = ax^3 + bx^2 + cx + d$. Bertha wins if the polynomial has at least two distinct real zeros (two roots of $f(x) = 0$); otherwise Adolf wins. Which player has a winning strategy?

5. Digit sum of the cube.

Some positive integers have the interesting property that if you cube the number and add up the (decimal) digits, you get the original number. For example, $8^3 = 512$, and $5+1+2 = 8$. Also, $17^3 = 4913$, and $4+9+1+3 = 17$. Are there infinitely many such integers? Defend your answer.

6. No real zeros.

Let

$$P(x) = x^{50} - 2x^{49} + 3x^{48} - 4x^{47} + \cdots - 50x + 51.$$

Prove that the equation $P(x) = 0$ has no real roots.

7. Not both rational.

The nonzero real numbers a and b satisfy

$$a^2b^2(a^2b^2 + 4) = 2(a^6 + b^6).$$

Prove that a and b are not both rational.

8. A global maximum.

For positive real numbers x and y, let $f(x,y) = \min\left\{x, \dfrac{y}{x^2+y^2}\right\}$. Show that there exist positive numbers x_0, y_0 such that $f(x,y) \leq f(x_0,y_0)$ for all $x > 0, y > 0$, and find this maximum value $f(x_0,y_0)$.

9. A set containing 2014, 0 and -2.

Let S be a set of integers containing the numbers 0 and 2014. Assume further that if r is an integer root of a polynomial with coefficients in S, then $r \in S$. Prove that $-2 \in S$.

10. An inequality.

Show that if $a > 0$ and n is an integer with $n > 3$, then

$$\frac{1 + a + a^2 + \cdots + a^n}{a^2 + a^3 + \cdots + a^{n-2}} \geq \frac{n+1}{n-3}.$$