

**SEVENTEENTH ANNUAL**  
**NORTH CENTRAL SECTION MAA**  
**TEAM CONTEST**

November 16, 2013,      9:00 a.m. to 12:00 noon

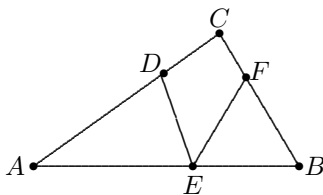
**NO BOOKS, NOTES, CALCULATORS, OR NON-TEAM-MEMBERS** may be consulted. **NO CELL PHONES OR OTHER ELECTRONIC DEVICES.**

**PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER.** Team identification and problem number should be given at the top of each sheet of paper submitted.

Each problem counts 10 points. Partial credit is given for significant but incomplete work. For full credit, answers must be fully justified. Have fun!

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**1. Measure of an angle.**



In the triangle  $ABC$  at the left, point  $D$  is on side  $AC$ ,  $E$  is on side  $AB$ , and  $F$  is on side  $BC$ . The measure of angle  $C$  is  $94^\circ$ . If  $AD = AE$  and  $BE = BF$ , find the measure of angle  $DEF$ .

**2. The 2013th term.**

An increasing sequence 3, 15, 24, ... is formed by the positive multiples of 3 that are one less than a perfect square. Find the 2013th term of the sequence.

**3. Probability of real roots.**

Two real numbers  $a$  and  $b$  are selected independently at random from the interval  $[0, 1]$ ; i.e., the interval has uniform probability distribution. What is the probability that the quadratic equation  $x^2 + ax + b = 0$  has real roots?

**4. Sum is 2013.**

Find all pairs  $(n, k)$  of positive integers such that the sum of the integers from  $n + 1$  through  $n + k$  inclusive is 2013.

**5. 20th derivative at 0.**

Let  $f(x) = (x + 2)^2 \cos 3x$ . Show that  $f^{(20)}(0)$ , the 20-th derivative at 0, is an integer, and express this integer as a product of prime powers.

**6. Predicted time remaining is one hour.**

An automobile on a 100 mile journey is equipped with a computer which at each instant predicts how much time remains until arriving at its destination. This prediction is made on the assumption that the average speed for the remainder of the trip will be the same as that for the part already completed. (Average speed over any segment is the number of miles traveled divided by the number of hours it took to travel that segment.) Thirty minutes after starting the journey the computer predicts that the remaining time will be one hour. Is it possible that for the next four hours this predicted remaining time will always be one hour? If so, (a) how many miles did the auto travel during these four hours? (b) What was the auto's speed one hour after departure?

**7. A periodic sequence.**

A sequence  $\{a_n\}$  of positive integers is defined recursively as follows: Let  $a_1$  be a positive integer, and for  $n \geq 1$ , let

$$a_{n+1} = \begin{cases} a_n/2 & \text{if } a_n \text{ is even;} \\ a_n + 2013 & \text{if } a_n \text{ is odd.} \end{cases}$$

Prove that the sequence is periodic from some point onward.

**8. A commutative operation.**

The set  $S$  is closed under an operation which we write multiplicatively: If  $x$  and  $y$  are in  $S$ , then  $xy \in S$ . The operation satisfies the following two axioms:

(A1) For all  $x, y \in S$ ,  $x(xy) = y$ .

(A2) For all  $x, y \in S$ ,  $(xy)y = x$ .

Prove that the operation is commutative. Justify each step carefully by referring to (A1) and (A2).

**9. Sum of interval lengths.**

For real  $x$  not in the set  $\{-1, -2, -3\}$ , let

$$f(x) = \frac{1}{x+1} + \frac{2}{x+2} + \frac{3}{x+3}$$

Find, with proof, the sum of the lengths of the intervals in which  $f(x) \geq 1$ .

**10. Sum of cubes greater than 40.**

The sum of several positive numbers is 10, and the sum of their squares is greater than 20. Prove that the sum of the cubes of these numbers is greater than 40.