SIXTEENTH ANNUAL
NORTH CENTRAL SECTION MAA
TEAM CONTEST

November 10, 2012, 9:00 a.m. to 12:00 noon

NO BOOKS, NOTES, CALCULATORS, COMPUTERS OR NON-TEAM-MEMBERS may be consulted. No cell phones or other communication devices.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team identification and problem number should be given at the top of each sheet of paper submitted.

Each problem counts 10 points. Partial credit is given for significant but incomplete work. For full credit, answers must be fully justified. But in some cases this may simply mean showing all work and reasoning. Have fun!

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1. A rational function of the medians.

Let ABC be any right triangle and let r, s and t be the lengths of the medians, with $r \geq s \geq t$. Show that

$$\frac{r^2 + s^2}{t^2} = 5.$$

2. A base 7 expansion.

The base 10 expansion of a certain rational number is $0.222\ldots$, or $0.\overline{2}$. Find the base 7 expansion of the same number. Explain your answer. (Your work may be enough to do so.)

3. Solve for x.

Find all positive solutions of the equation

$$x^{(x^3)} = 3.$$

Some trial and error may be necessary, but you must show that you have all the solutions.

4. A 2012 trigonometric identity.

Prove that if $\sin x \cos x \neq 0$, then

$$\frac{\sin x}{\cos x} + \frac{\sin 2x}{\cos^2 x} + \cdots + \frac{\sin 2011x}{\cos^{2011} x} = \cot x - \frac{\cos 2012x}{\sin x \cos^{2011} x}.$$

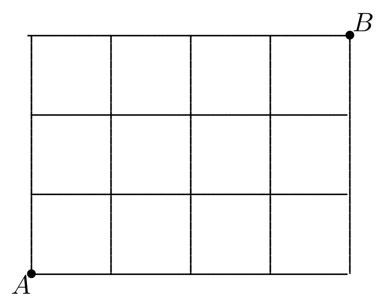
5. A polynomial extrapolation.

The monic polynomial $f(x)$ has degree 9, and $f(n) = n$ for each integer n, $1 \le n \le 9$. Find $f(10)$, and justify your answer. ("Monic" means with leading coefficient 1.)

6. A 2012 stones game.

Two players, A and B, play the following game. There is initially a pile of 2012 stones, and the players alternate in removing some stones. Player A goes first, and chooses a positive divisor of 2012 and removes that many stones from the pile. Then B chooses a positive divisor of the number of stones remaining, and removes that many. They continue in this manner, and the player who takes the last stone loses. Show that there is a winning strategy for one of the players, and describe that strategy, making clear that it wins.

7. Probability that they meet.



The figure at the right represents a uniform 3 by 4 grid of streets. Adolf will walk from A to B and Bertha will walk from B to A, taking paths of minimum length, namely 7 blocks. They start at the same time and walk at the same constant speed. At each intersection where there are two options for the direction to take, one is chosen at random (so, each has probability 1/2). What is the probability that they meet along the way? (That "they meet" means that at some time they are both at the same place.)

8. Trading denominators to make the sum 2012.

Do there exist positive integers a, b, c, d such that $\frac{a}{b} + \frac{c}{d} = 1$ and $\frac{a}{d} + \frac{c}{b} = 2012$?

9. Divisors of $(n+1)!$.

Determine all positive integers n such that $(n+1)!$ is divisible by $1! + 2! + \cdots + n!$.

10. A sequence dense in (0,1).

Let S be the set of rational numbers in the interval (0,1). Show that it is possible to arrange the elements of S in a sequence $a_1, a_2, a_3, \ldots$ such that (i) each element of S appears exactly once in the sequence, and (ii) the corresponding "exponentialized" sequence $a_1, a_2^2, a_3^3, \ldots$ is also dense in $(0, 1)$.