FIFTEENTH ANNUAL
NORTH CENTRAL SECTION MAA
TEAM CONTEST

November 12, 2011, 9:00 a.m. to 12:00 noon

NO BOOKS, NOTES, CALCULATORS, COMPUTERS OR NON-TEAM-MEMBERS may be consulted. No cell phones or other communication devices.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team identification and problem number should be given at the top of each sheet of paper submitted.

Each problem counts 10 points. Partial credit for significant but incomplete work. For full credit, answers must be fully justified. But in some cases this may simply mean showing all work and reasoning. Have fun!

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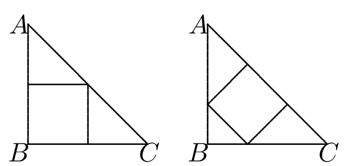
1. Units digit of S_{2011}.

Let S_n be the sum of the squares of the first n positive odd integers. Find, with proof, the units digit (in base 10) of S_{2011}.

2. Area of a square.



At the left are two copies of the same isosceles right triangle ABC, each with an inscribed square. If the first (leftmost) inscribed square has area S, what is the area of the other inscribed square? Justify your answer.

3. Speshul integers.

A positive integer k is called "speshul" if there exist positive integers m and n such that

$$\frac{mn+1}{m+n} = k.$$

Find all speshul positive integers.

4. Divisor and remainder.

Determine all pairs (d, r) of positive integers with the property that when each of the numbers 904, 1259 and 2040 is divided by d, the same remainder r (with $0 < r < d$) occurs.

5. Bigger than 2011^2?.

Determine whether there exist non-negative numbers x and y such that $x + y = 2011$ and $x^2 + \sqrt{y} > 2011^2$.

6. From 11 to 2011.

The positive integer n can be replaced by ab, if a and b are positive integers with $a + b = n$. Is it possible starting from 11 to obtain 2011 in a finite sequence of such replacements?

7. Not a perfect square.

For each positive integer n, let $a_n = \lfloor (n + \sqrt{19})^2 + 2n + \sqrt{99} \rfloor$. (As usual, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.) Show that a_n is never the square of an integer.

8. $P(11) = 2011$.

The coefficients a_j in the polynomial $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ are integers satisfying $0 \le a_j \le 10$ and $a_n > 0$. If $P(1) = 21$ and $P(11) = 2011$, determine, with proof, the value of $P(6)$.

9. Fractional part of $\sqrt{n}$.

Let n be a positive integer which is not a perfect square. Let x_n be the fractional part of $\sqrt{n}$; i.e., $x_n = \sqrt{n} - \lfloor \sqrt{n} \rfloor$, where $\lfloor u \rfloor$ denotes as usual the greatest integer less than or equal to u. Prove that

$$x_n + \frac{1}{2n} < 1.$$

10. Numerator divisible by 2011.

Let p and q be positive integers such that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{1339} - \frac{1}{1340} = \frac{p}{q}.$$

Given that 2011 is prime, show that p is divisible by 2011.