TWENTY-FIFTH ANNUAL

NORTH CENTRAL SECTION MAA

HEUER MEMORIAL TEAM CONTEST

November 11, 2023, 9:00 a.m. to 12:00 noon

NO BOOKS, NOTES, CALCULATORS, OR NON-TEAM-MEMBERS MAY BE CONSULTED. NO CELL PHONES OR OTHER ELECTRONIC DEVICES.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team name and problem number should be given at the top of each sheet of paper submitted. Each problem counts 10 points. Partial credit is given for significant but incomplete work. For full credit, answers must be fully justified. Have fun!

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1. Area between two Curves

If m is a positive real number and the area enclosed by the graphs of $y = mx$ and $y = x^2$ is 9, determine the value of m.

2. Sum of Legs in a Right Triangle

A right triangle has legs of length a, b, and hypotenuse of length c. Prove that

$$a + b \leq c\sqrt{2}.$$

3. Log Equation

Determine all possible real values of x that satisfy the following equation:

$$289^x = 289^{\log_7(2)} + 2^{\log_7(289)} - 2023^{\log_7(2)} + 1.$$

4. Sum of Squares of Rational Numbers

Let a, b, and c be three distinct rational numbers. Prove that

$$\sqrt{\frac{1}{(b-c)^2} + \frac{1}{(c-a)^2} + \frac{1}{(a-b)^2}}$$

is a rational number.

5. A Tale of Two Squares

The vertices of a square all lie on a circle C. Two adjacent vertices of another square lie on circle C while the other two lie on one of its diameters. Find the ratio of the area of the smaller square to the area of the larger square.

6. Integral Equality

Let f be a function continuous on $[0, \pi]$. Show that

$$\int_0^\pi x f(\sin x)\, dx = \frac{\pi}{2} \int_0^\pi f(\sin x)\, dx.$$

7. SUV in a Parking Lot

A parking lot has 16 parking cells side by side in a row. Starting with an empty lot, twelve cars enter the lot, each driver selecting one of the vacant cells at random. Then an oversize SUV which requires two adjacent cells to park enters. What is the probability that there is such a double cell available?

8. Convergent Series

Prove that if a is a real number, the series

$$\sum_{n=0}^{\infty} \sin(\pi\sqrt{n^2 + a^2})$$

converges.

9. Difference of Cubes

Prove that if the difference of the cubes of two successive integers is a square, then it is the square of the sum of two successive squares. For example: $8^3 - 7^3 = 169 = 13^2 = (2^2 + 3^2)^2$.

10. Simple Function

Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ is such that for all x and y in $\mathbb{R}$

$$f(x) + f(y) = f(f(x) \cdot f(y)).$$

Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.