TWENTY-FOURTH ANNUAL

NORTH CENTRAL SECTION MAA

HEUER MEMORIAL TEAM CONTEST

Solutions

November 11, 2023

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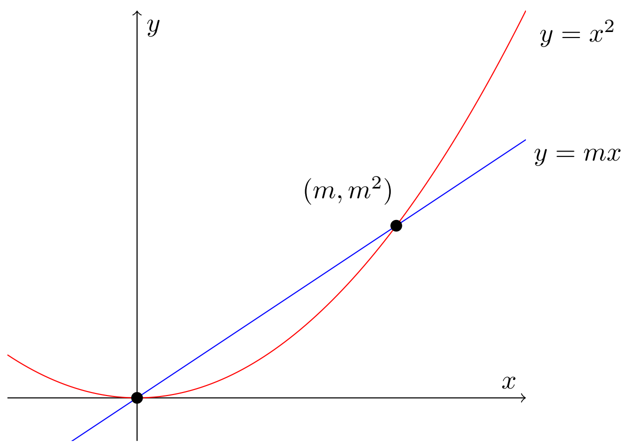
1. Area between two Curves

If m is a positive real number and the area enclosed by the graphs of $y = mx$ and $y = x^2$ is 9, determine the value of m.

Solution. The value of m is $\sqrt[3]{54} = 3\sqrt[3]{2}$.



The graphs of the two curves intersect where x satisfies the equation:

$$mx = x^2$$

that is to say, either $x = 0$ or $x = m$. More precisely, the two intersection points are $(0, 0)$ and (m, m^2), with the graph of $y = x^2$ below the graph of mx in the interval between.
To determine the area between the two curves we evaluate the following definite integral:

$$\int_0^m (mx - x^2)\, dx = \left[\frac{mx^2}{2} - \frac{x^3}{3}\right]_0^m = \frac{m^3}{2} - \frac{m^3}{3} = \frac{m^3}{6}.$$

We need to determine the value of m that satisfies the following equality:

$$\frac{m^3}{6} = 9,$$

that is to say $m^3 = 54$ and $m = \sqrt[3]{54} = 3\sqrt[3]{2}$.

2. Sum of Legs in a Right Triangle

A right triangle has legs of length a, b, and hypotenuse of length c. Prove that

$$a + b \leq c\sqrt{2}.$$

Solution I. Since a, b, and c are all positive real numbers, the inequality is equivalent to:

$$2c^2 \geq (a+b)^2.$$

Now, because a, b, c are the lengths of the legs and the hypothenuse of a right triangle, we have that $c^2 = a^2 + b^2$, so we want to prove:

$$2a^2 + 2b^2 \geq a^2 + 2ab + b^2.$$

This is equivalent to:

$$a^2 - 2ab + b^2 \geq 0,$$

that is to say:

$$(a-b)^2 \geq 0.$$

Since the square of a real number is non-negative, the inequality holds.

Solution II. Let θ be the angle between the leg of length b and the hypotenuse of length c. Then, from basic right triangle trigonometry, we get:

$$a + b = c(\sin(\theta) + \cos(\theta)).$$

We now define the function:

$$f(\theta) := c(\sin(\theta) + \cos(\theta))$$

where the domain of the variable θ is $[0, \pi/2]$. Our goal is to maximize the function $f(\theta)$ over the given interval. From basic calculus results, we have that the maximum of $f(\theta)$ occurs either at an endpoint of the interval, or at a critical point, that is to say a point such that $f'(\theta) = 0$. We get:

$$f'(\theta) = c(\cos(\theta) - \sin(\theta)),$$

which is 0 on the interval $[0, \pi/2]$ if and only if $\theta = \pi/4$. We now evaluate the function at the candidate points for the maximum:

$f(0) = c(0+1) = c$

$f(\pi/4) = c\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) = c\sqrt{2}$

$f(\pi/2) = c(1+0) = c.$

The largest of these values is $c\sqrt{2}$, therefore we get:

$$a + b \leq c\sqrt{2}.$$

3. Log Equation

Determine all possible real values of x that satisfy the following equation:

$$289^x = 289^{\log_7(2)} + 2^{\log_7(289)} - 2023^{\log_7(2)} + 1.$$

Solution. The solution is $x = 0$.
First we show that $289^{\log_7(2)} = 2^{\log_7(289)}$. From the definition of logarithm, we have

$$289 = 7^{\log_7(289)} \text{ and } 2 = 7^{\log_7(2)}.$$

Therefore we have:

$$289^{\log_7(2)} = \left(7^{\log_7(289)}\right)^{\log_7(2)} = \left(7^{\log_7(2)}\right)^{\log_7(289)} = 2^{\log_7(289)}.$$

By applying the identity above and properties of logarithms and exponents to the original equation, we get:

$$\begin{aligned} 289^x &= 289^{\log_7(2)} + 2^{\log_7(289)} - 2023^{\log_7(2)} + 1 = \\ &= 289^{\log_7(2)} + 289^{\log_7(2)} - (7 \cdot 289)^{\log_7(2)} + 1 = \\ &= 2 \cdot 289^{\log_7(2)} - \left(7^{\log_7(2)}\right) 289^{\log_7(2)} + 1 = \\ &= 2 \cdot 289^{\log_7(2)} - 2 \cdot 289^{\log_7(2)} + 1 = \\ &= 1. \end{aligned}$$

In conclusion, we need to solve the equation $289^x = 1$ which has only one solution among the real numbers: $x = 0$.

4. Sum of Squares of Rational Numbers

Let a, b, and c be three distinct rational numbers. Prove that

$$\sqrt{\frac{1}{(b-c)^2} + \frac{1}{(c-a)^2} + \frac{1}{(a-b)^2}}$$

is a rational number.

Solution I. Let $x = b - c$, $y = c - a$, and $z = a - b$. Then $x + y + z = 0$, $xyz \neq 0$. and we want to show that

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{y^2z^2 + x^2z^2 + x^2y^2}{x^2y^2z^2}$$

is the square of a rational number. We only need to prove that the numerator is the square of a rational number. Since we have the equality $z^2 = (x + y)^2$, the numerator becomes:

$$y^2(x + y)^2 + x^2(x + y)^2 + x^2y^2,$$

which is equal to

$$x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + y^4 = (x^2 + xy + y^2)^2.$$

Solution II. Let

$$x = \frac{1}{b-c}, \quad y = \frac{1}{c-a}, \quad \frac{1}{a-b}.$$

Then we have:

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0,$$

and we want to show that $x^2 + y^2 + z^2$ is the square of a rational number. Now,

$$x^2 + y^2 + z^2 = (x + y + z)^2 - 2(xy + yz + xz).$$

Moreover,

$$xy + yz + xz = xyz\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 0,$$

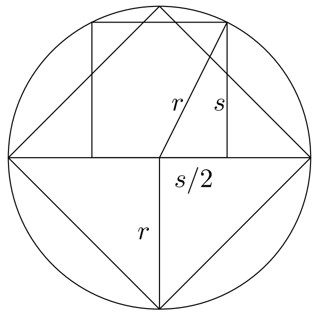
so

$$x^2 + y^2 + z^2 = (x + y + z)^2.$$

5. A Tale of Two Squares

The vertices of a square all lie on a circle C. Two adjacent vertices of another square lie on circle C while the other two lie on one of its diameters. Find the ratio of the area of the smaller square to the area of the larger square.

Solution. The ratio is $2/5$.



With reference to the figure above, let r be the radius of the circle and let s be the side length of the smaller square. Then the area of the larger square is $2r^2$ and the area of the smaller square is s^2. From the right triangle with sides $s/2$, s, and r we have:

$$r^2 = s^2 + \left(\frac{s}{2}\right)^2 = \frac{5}{4}s^2.$$

Consequently, we have:

$$s^2 = \frac{4}{5}r^2$$

and the desired ratio is:

$$\frac{s^2}{2r^2} = \frac{(4/5)r^2}{2r^2} = \frac{2}{5}.$$

6. Integral Equality

Let f be a function continuous on $[0, \pi]$. Show that

$$\int_0^\pi x f(\sin x)\, dx = \frac{\pi}{2} \int_0^\pi f(\sin x)\, dx.$$

Solution. The problem is equivalent to show the following equality:

$$I := \int_0^\pi \left(x - \frac{\pi}{2}\right) f(\sin x)\, dx = 0.$$

Make the substitution $u = x - \pi/2$ to get

$$\int_{-\pi/2}^{\pi/2} u f\left(\sin\left(u + \frac{\pi}{2}\right)\right)\, du.$$

Since

$$\sin\left(\frac{\pi}{2} - u\right) = \sin\left(\frac{\pi}{2} + u\right),$$

the integrand is an odd function of u, and therefore $I = 0$.

7. SUV in a Parking Lot

A parking lot has 16 parking cells side by side in a row. Starting with an empty lot, twelve cars enter the lot, each driver selecting one of the vacant cells at random. Then an oversize SUV which requires two adjacent cells to park enters. What is the probability that there is such a double cell available?

Solution. The probability is $\dfrac{1105}{1820} = \dfrac{17}{28}$.

The problem is equivalent to choosing 4 of the 16 cells at random and asking for the probability that some two of the four are adjacent.
We first calculate the probability of the complementary event, that no two of the four remaining spots are adjacent.
If we append a 17th occupied cell, then each empty cell is followed by an occupied one, and the problem is equivalent to counting the number of combinations of 4 empty-occupied pairs (which can be thought to be compressed into a single cell) and 9 occupied singletons, which is $\binom{13}{4} = 715$.
There are $\binom{16}{4} = 1820$ sets of four cells in all, so the probability that that no double cell is unoccupied is $\dfrac{715}{1820}$, and the probability that an occupied cell is available is

$$1 - \frac{715}{1820} = \frac{1105}{1820} = \frac{17}{28}.$$

8. Convergent Series

Prove that if a is a real number, the series

$$\sum_{n=0}^{\infty} \sin(\pi\sqrt{n^2+a^2})$$

converges.

Solution. We use the identity

$$(\sqrt{n^2+a^2}-n)(\sqrt{n^2+a^2}+n) = a^2$$

to obtain

$$\sqrt{n^2+a^2} = \frac{a^2}{\sqrt{n^2+a^2}+n} + n.$$

Then we have

$$\sin(\pi\sqrt{n^2+a^2}) = \sin\left(\frac{\pi a^2}{\sqrt{n^2+a^2}+n} + \pi n\right) = (-1)^n \sin\left(\frac{\pi a^2}{\sqrt{n^2+a^2}+n}\right).$$

The terms of the series are therefore of alternating sign and decreasing in magnitude to 0. By the alternating series theorem the series is convergent.

9. Difference of Cubes

Prove that if the difference of the cubes of two successive integers is a square, then it is the square of the sum of two successive squares. For example: $8^3 - 7^3 = 169 = 13^2 = (2^2+3^2)^2$.

Solution. Assume that

$$(n+1)^3 - n^3 = 3n^2 + 3n + 1 = m^2$$

for some integers n and m. We want to show that

$$m^2 = r^2 + (r+1)^2$$

for some integer r. By multiplying the first equation by 4 we get:

$$4m^2 = 12n^2 + 12n + 4$$

and we have

$$(2m-1)(2m+1) = 4m^2 - 1 = 12n^2 + 12n + 3 = 3(4n^2 + 4n + 1) = 3(2n+1)^2.$$

Since $2m-1$ and $2m+1$ are two odd integers whose difference is 2, they must be relatively prime. This implies that we have the following two possibilities.

(a) $2m - 1 = 3p^2$ and $2m + 1 = q^2$ for some integers a and b, or

(b) $2m - 1 = p^2$ and $2m + 1 = 3q^2$ for some integers a and b.

In the first case we get $3p^2 + 2 = q^2$, which is impossible because 2 is not a square mod 3. Therefore, we must have case (b). In particular, p must be odd, that is to say $p = 2r + 1$ for some integer r. Then we have:

$$2m = p^2 + 1 = 4r^2 + 4r + 2$$

and

$$m = 2r^2 + 2r + 1 = r^2 + (r+1)^2$$

as desired.

10. Simple Function

Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ is such that for all x and y in $\mathbb{R}$

$$f(x) + f(y) = f(f(x) \cdot f(y)).$$

Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

Solution. Let x_0 be a real number and let us define $c := f(x_0)$. Taking $x = y = x_0$, we have:

$$f(x_0) + f(x_0) = f(f(x_0) \cdot f(x_0))$$

that is to say:

$$2c = f(c^2).$$

If we take $x = y = c^2$ we get:

$$\begin{aligned} f(c^2) + f(c^2) &= f(f(c^2) \cdot f(c^2)) \\ 2c + 2c &= f(2c \cdot 2c) \\ 4c &= f(4c^2). \end{aligned}$$

Next by choosing $x = x_0$ and $y = 4c^2$ we get

$$\begin{aligned} f(x_0) + f(4c^2) &= f(f(x_0) \cdot f(4c^2)) \\ c + 4c &= f(c \cdot 4c) \\ 5c &= f(4c^2). \end{aligned}$$

Consequently, we must have $4c = 5c$ which implies that $c = 0$. But x_0 is an arbitrary real number, so for every real number x we have $f(x) = 0$.