TWENTY-FOURTH ANNUAL

NORTH CENTRAL SECTION MAA

HEUER MEMORIAL TEAM CONTEST

Solutions

November 12, 2022

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1. Necklace with 21 Diamonds

A necklace has 21 diamonds. The middle one is the largest, and they taper off in value toward each end. Beginning from one end, each successive diamond is worth \$100 more than the preceding one, until the middle is reached. Beginning from the other end, each one is worth \$150 more than its predecessor. The total value of the diamonds in the necklace is \$47,150. What is the value of the middle diamond?

Solution. The value of the middle diamond is $\$2,900$. Let x be the dollar value of the middle diamond. Then the values of its ten neighbors on one side are $x - 100$, $x - 200$, $\ldots$, $x - 1000$, and those on the other side are $x - 150$, $x - 300$, $\ldots$, $x - 1500$. By pairing each one on one side with the corresponding one on the other side we may express the total value as

$$\begin{aligned} 47150 &= x + (2x - 250) + (2x - 500) + \cdots + (2x - 2500) \\ &= 21x - 250(1 + 2 + 3 + \cdots + 10) \\ &= 21x - 250 \cdot 55 \\ &= 21x - 13750, \end{aligned}$$

so $21x = 47150 + 13750 = 60900$, and $x = 2900$.

2. Square with Perpendicular to a Diagonal

In the square $ABCD$, the line from a point E on side CD to a point G on side BC is perpendicular to the diagonal AC and intersects it at F. If $|AF| = |EG| = 20$, determine $|DE|$. Here, the notation $|DE|$ denotes the length of side DE.

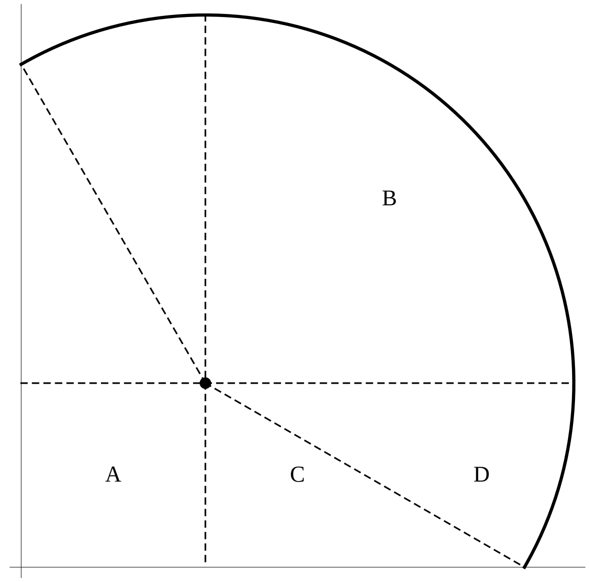
Solution. The length $|DE|$ is $5\sqrt{2}$. By the statement of the problem we see that $\angle EFC = \angle CFG = 90°$ and $\angle FCE = \angle FCG = 45°$. Therefore EFC and CFG are congruent right isosceles triangles with $|FC| = |EF| = 20/2 = 10$. Then $|EC| = 10\sqrt{2}$, $|AC| = 20 + 10 = 30$, and $|DC| = 30/\sqrt{2} = 15\sqrt{2}$, from which it follows that $|DE| = |DC| - |EC| = 5\sqrt{2}$.

3. Area of a Region

Determine the area of the region S defined by

$$S = \{(x, y) : (|x| - 1)^2 + (|y| - 1)^2 \leq 4\}.$$

Solution. The region is symmetric across the x- and y-axes due to the absolute values, so it suffices to consider the part of the region in the first quadrant – which is the part of the disk $(x - 1)^2 + (y - 1)^2 \leq 4$ in the first quadrant.



This can be subdivided into simpler regions as shown. Region A is a square, with Area$(A) = 1$. Region B is a quarter circle, with Area$(B) = (1/4) \cdot \pi \cdot 2^2 = \pi$.

Region C is a right triangle with height 1 and hypotenuse 2, hence Area$(C) = \sqrt{3}/2$.

Region D is a sector with radius 2 and central angle $\pi/6$ hence Area$(D) = (1/2)(\pi/6) \cdot 2^2 = \pi/3$.

Adding up, the area in the first quadrant is

$$1 + \pi + 2\left(\sqrt{3}/2\right) + 2\left(\pi/3\right) = 1 + \sqrt{3} + \frac{5\pi}{3},$$

hence, by the four-fold symmetry,

$$\text{Area}(S) = 4 + 4\sqrt{3} + \frac{20\pi}{3}.$$

4. Matrix Involution

Show that there are infinitely many 3×3 matrices with integer entries such that $M^2 = I$, where I is the 3×3 identity matrix.

Solution. Let

$$M = \begin{bmatrix} 1 & 0 & 0 \\ k & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where k is any integer. We have:

$$M^2 = \begin{bmatrix} 1 & 0 & 0 \\ k & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ k & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 & 0 & 0 \\ k - k & (-1) \cdot (-1) & 0 \\ 0 & 0 & 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

5. Evaluating Polynomials

The monic polynomial $f(x)$ has degree 2021, and $f(n) = n$ for each integer n, $1 \le n \le 2021$. Find $f(2022)$, and justify your answer.

Solution. The answer is $2022 + 2021!$
Since $f(x)$ is monic of degree 2021, the polynomial $f(x) - x$ is also monic and of degree 2021. By hypothesis, the polynomial $f(x) - x$ has 2021 zeros, namely the integers from 1 to 2021. Thanks to the fundamental theorem of algebra, and the fact that $f(x) - x$ is monic we can write:

$$f(x) - x = \prod_{i=1}^{2021}(x-i) = (x-1)(x-2)\ldots(x-2020)(x-2021).$$

Therefore, we have the identity:

$$f(x) = x + \prod_{i=1}^{2021}(x-i).$$

Now we evaluate for $x = 2022$:

$$f(2022) = 2022 + \prod_{i=1}^{2021}(2022-i) = 2022 + 2021!$$

6. Sums of Reciprocals

Let x, y, z be positive real numbers.

(a) If $x + y + z \geq 3$, does it follow that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq 3$?

(b) If $x + y + z \leq 3$, does it follow that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 3$?

Solution.

(a) No; for example, $(x, y, z) = (1, 2, 1/3)$ has $x + y + z > 3$ and $1/x + 1/y + 1/z = 9/2$. In fact, the sum of reciprocals can be made arbitrarily large by taking $x = 1$, $y = 2$, and z arbitrarily close to zero.

(b) Yes. First note that, if α is a positive real number, then $(\alpha - 1)^2 \geq 0$ implies $\alpha^2 + 1 \geq 2\alpha$ and dividing by α gives $\alpha + (1/\alpha) \geq 2$. Now, suppose x, y, and z are positive, with $x + y + z \leq 3$. Then we have

$$x + y + z + \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = \left(x + \frac{1}{x}\right) + \left(y + \frac{1}{y}\right) + \left(z + \frac{1}{z}\right) \geq 2 + 2 + 2 = 6,$$

and so

$$\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geq 6 - (x + y + z) \geq 3.$$

7. Integer Triangles

Find all (nondegenerate) triangles with integral sides, one side equal to 10, and the cosine of an adjacent angle equal to $-1/5$.

Solution. There are three such triangles, namely $\{10, 25, 21\}, \{10, 14, 8\}, \{10, 11, 13\}$.
Let the other two sides be a and b, with a opposite the angle with cosine equal to $-1/5$. By the law of cosines we have:

$$a^2 = b^2 + 10^2 - 2b \cdot 10(-1/5) = b^2 + 4b + 100.$$

This implies

$$\begin{aligned} a^2 - b^2 - 4b &= 100, \\ a^2 - (b+2)^2 &= 100 - 4, \\ (a+b+2)(a-b-2) &= 96. \end{aligned}$$

The two factors on the left in the last equation have the same parity, so both must be even. Moreover, since the larger factor $a + b + 2$ is positive, the smaller factor $a - b - 2$ must be positive as well. The possible values for the smaller factor $a-b-2$ are $2, 4, 6$, and 8. The corresponding values for the larger factor $a+b+2$ are $48, 24, 16$, and 12. Solving the four resulting pairs of simultaneous equations we get

$$(a, b) = (25, 21), (14, 8), (11, 3), \text{ and } (10, 0).$$

The last corresponds to a degenerate triangle, so there are exactly three triangles $\{10, 25, 21\}, \{10, 14, 8\}$, and $\{10, 11, 13\}$.

8. Bounded Series

Prove that for every integer $n > 1$,

$$\frac{1}{3^2} + \frac{1}{5^2} + \ldots + \frac{1}{(2n+1)^2} < \frac{1}{4}.$$

Solution. The following inequality is satisfied for every positive integer k:

$$k^2 > k^2 - 1 = (k-1)(k+1).$$

Consequently, we have:

$$\begin{aligned} \frac{1}{3^2} + \frac{1}{5^2} + \ldots + \frac{1}{(2n+1)^2} &< \frac{1}{2 \cdot 4} + \frac{1}{4 \cdot 6} + \ldots \frac{1}{2n(2n+2)} \\ &= \frac{1}{2}\left(\left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{2n} - \frac{1}{2n+2}\right)\right), \end{aligned}$$

where the last identity comes from the fact that for every integer k:

$$\frac{1}{(k-1)(k+1)} = \frac{1}{2}\left(\frac{1}{k-1} - \frac{1}{k+1}\right).$$

After simplifying (or "telescoping"), we get:

$$\cdots = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2n+2}\right) < \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

9. Same Difference

Let $a_1, a_2, \ldots, a_{2n}$ be $2n$ distinct integers with $n > 1$ and $0 < a_i \leq n^2$ for each i. Prove that some three of the differences $a_i - a_j$ (with $i \neq j$) are equal.

Solution. We assume without loss of generality, that

$$1 \leq a_1 < a_2 < \cdots < a_{2n} \leq n^2.$$

Consider the $2n-1$ differences

$$a_{i+1} - a_i, \quad 1 \leq i \leq 2n-1.$$

We want to show that in fact some three of these differences must be equal. If no three of these are equal, their sum is at least:

$$1 + 1 + 2 + 2 + \cdots + (n-1) + (n-1) + n = 2\left(\sum_{i=1}^{n-1} i\right) + n = 2 \cdot \frac{n(n-1)}{2} + n = n^2.$$

But this contradicts the fact that their sum is $a_{2n} - a_1 \leq n^2 - 1$. Therefore, some three of the differences $a_{i+1} - a_i$ are equal.

10. Integral Solutions

Consider the system of equations

$$x+y+z=3, \tag{1}$$
$$x^5+y^5+z^5=33, \tag{2}$$

Find all of its solutions in integers (with justification) or show that no such solutions exist.

Solution. The are 6 integer solutions $(x,y,z)=(0,1,2)$ and its permutations. If x_0, y_0, and z_0 are integers satisfying the equations, then they also satisfy $(x+y+z)^5-(x^5+y^5+z^5)=3^5-33=210$. The polynomial in the left-hand side, considered as a polynomial in x, vanishes for $x=-y$, so $x+y$ is a factor. By symmetry, $x+z$ and $y+z$ also are factors. From Pascal's triangle it follows that 5 is also a factor. Thus, for some polynomial $Q(x,y,z)$ we have, using (1),

$$5(3-x_0)(3-y_0)(3-z_0)Q(x_0,y_0,z_0)=210=1\times 2\times 3\times 5\times 7,$$

or

$$(3-x_0)(3-y_0)(3-z_0)Q(x_0,y_0,z_0)=210=1\times 2\times 3\times 7.$$

From (1) we have that either all of x_0, y_0, z_0 are odd, or two are even and one is odd. In the first case, all of $(3-x_0)$, $(3-y_0)$, $(3-z_0)$ are even, which is impossible based on the prime factorization of 210, so two of them have to be odd and one even. Moreover, from (1) we see that

$$(3-x_0)+(3-y_0)+(3-z_0)=6. \tag{3}$$

Assuming $3-x_0$ is the even term, we have $3-x_0$ is one of ± 2, ± 6, ± 14, or ± 42. The last 4 possibilities can be excluded since they won't allow (3) to be satisfied with the remaining factors of 210 (e.g., if $3-x_0=14$, that leaves $3-y_0=\pm 3$ and $3-z_0=\pm 1$ or the other way around and (3) can't be satisfied; similarly for the other three possibilities). The remaining possibilities with $3-x_0$ equal to ± 2 or ± 6 can be easily checked to be

$$2+1+3=6, \quad 2-3+7=6, \quad -2+1+7=6, \quad 6+1-1=6.$$

These lead to (up to permutations of the order)

$$\begin{gathered} x_0=1,\ y_0=2,\ z_0=0, \\ x_0=1,\ y_0=6,\ z_0=-4, \\ x_0=5,\ y_0=2,\ z_0=-4, \\ x_0=-3,\ y_0=2,\ z_0=4. \end{gathered}$$

Of these the first one clearly satisfies (2) since $1^5+2^5+0^5=33$. The second possibility would yield $1^5+(6^5-4^5)=1+(6-4)(6^4+6^3\cdot 4+6^2\cdot 4^2+6\cdot 4^3+4^4)>1+2\cdot 5\cdot 4^4>33$ and similarly for the last two. Hence only the first possibility (and its permutations) are solutions of our system of equations.