TWENTY-FOURTH ANNUAL

NORTH CENTRAL SECTION MAA

HEUER MEMORIAL TEAM CONTEST

November 12, 2022, 9:00 a.m. to 12:00 noon

NO BOOKS, NOTES, CALCULATORS, OR NON-TEAM-MEMBERS MAY BE CONSULTED. NO CELL PHONES OR OTHER ELECTRONIC DEVICES.

PLEASE BEGIN EACH PROBLEM ON A NEW SHEET OF PAPER. Team name and problem number should be given at the top of each sheet of paper submitted. Each problem counts 10 points. Partial credit is given for significant but incomplete work. For full credit, answers must be fully justified. Have fun!

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1. Necklace with 21 Diamonds

A necklace has 21 diamonds. The middle one is the largest, and they taper off in value toward each end. Beginning from one end, each successive diamond is worth \$100 more than the preceding one, until the middle is reached. Beginning from the other end, each one is worth \$150 more than its predecessor. The total value of the diamonds in the necklace is \$47,150. What is the value of the middle diamond?

2. Square with Perpendicular to a Diagonal

In the square $ABCD$, the line from a point E on side CD to a point G on side BC is perpendicular to the diagonal AC and intersects it at F. If $|AF| = |EG| = 20$, determine $|DE|$. Here, the notation $|DE|$ denotes the length of side DE.

3. Area of a Region

Determine the area of the region S defined by

$$S = \{(x, y) : (|x| - 1)^2 + (|y| - 1)^2 \leq 4\}.$$

4. Matrix Involution

Show that there are infinitely many 3×3 matrices M with integer entries such that $M^2 = I$, where I is the 3×3 identity matrix.

5. Evaluating Polynomials

The monic polynomial $f(x)$ has degree 2021, and $f(n) = n$ for each integer n, $1 \leq n \leq 2021$. Find $f(2022)$, and justify your answer.

6. Sums of Reciprocals

Let x, y, z be positive real numbers.

(a) If $x + y + z \geq 3$, does it follow that $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} \leq 3$?

(b) If $x + y + z \leq 3$, does it follow that $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} \geq 3$?

7. Integer Triangles

Find all (nondegenerate) triangles with integral sides, one side equal to 10, and the cosine of an adjacent angle equal to $-1/5$.

8. A Bound on Partial Sums

Prove that for every integer $n > 1$,

$$\frac{1}{3^2} + \frac{1}{5^2} + \ldots + \frac{1}{(2n+1)^2} < \frac{1}{4}.$$

9. Same Difference

Let $a_1, a_2, \ldots, a_{2n}$ be $2n$ distinct integers with $n > 1$ and $0 < a_i \leq n^2$ for each i. Prove that some three of the differences $a_i - a_j$ (with $i \neq j$) are equal.

10. Integral Solutions

Consider the system of equations

$$x + y + z = 3,$$
$$x^5 + y^5 + z^5 = 33.$$

Find all of its solutions in integers (with justification) or show that no such solutions exist.